CONSTRUCTION OF $\mathbb{R}$ FROM $\mathbb{Q}$

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1. Problem 1

Read the Appendix at the end of Chapter 1 (page 17) of Rudin's book about constructing $\mathbb{R}$ from $\mathbb{Q}$.

1.1. Summarize the nine steps and what they are proving.

(1) **Step 1** The first step to Rudin's proof about constructing $\mathbb{R}$ from $\mathbb{Q}$ defines "cuts" as subsets of the rationals with certain properties. The simplest of these properties is that a "cut", $\alpha \neq \emptyset$ and that $\alpha \neq \mathbb{Q}$. The next property says that if there is some element $p \in \alpha$ and another element $q \in \mathbb{Q}$ and we know that $p < q$, then that implies that $p \in \alpha$. This property fills the cut with the rationals that are less than p. The final property is if $p \in \alpha$, then $p < r$ for some $r \in \alpha$. This means that there is no largest number in the cut.

(2) **Step 2** In the next step, Rudin defines $\alpha < \beta$ to mean that α is a proper subset of β. Rudin then checks that this satisfies the requirements of an ordered set. Definition 1.5 says than an order on a set is a relation defined as one of the following: $x = y$, $x < y$, or $y < x$, as well as transitivity. Additionally, by defining $\alpha < \beta$ as a proper subset, it means that $\alpha \neq \beta$. Rudin successfully shows that $\mathbb{R}$ is an ordered set, which is crucial for the rest of the proof because it allows for ordering of elements and analysis of the next and previous elements.

(3) **Step 3** Up to this point, Rudin has only come up with a neat way of rewriting the rational numbers. The cuts so far include everything up until the irrational number. Each set so far is a set of rational numbers. Rudin proves in this step that the supremum of $\mathbb{R}$ is defined as $\gamma = \sup A$ where A is a non-empty subset of $\mathbb{R}$ that is bounded above and γ is defined by the union of all cuts in the set A. This means that the cuts have a least upper bound. Now, we know that $\mathbb{R}$ is ordered and there is a supremum.

(4) **Step 4** Now that the sets are ordered and there exists a least upper bound, cuts represent real numbers from the rational numbers. The next step Rudin takes to construct the real numbers from the rationals is to show that the axioms of addition hold. One of the more difficult aspects of this part of the proof is how to deal with 0 and he does this by defining 0^* to be the set of all negative rational numbers, namely $0^* = \{p \in \mathbb{Q} | p < 0\}$. He defines two cuts: $\alpha, \beta \in \mathbb{R}$ and defines $\alpha + \beta$ to be all sums $r + s$ where $r \in \alpha$ and $s \in \beta$. In this case, 0^* acts as 0 which allows Rudin to conclude that $\alpha + \beta = 0^*$.

(5) **Step 5** In this step, Rudin proves that if $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$. He proves this using the previous step where he defines addition.

(6) **Step 6** This step is similar to step 4, where he defines addition, but instead he defines multiplication. The problematic part of the definition again arises from the identity element

1 and the fact that the product of two negative rationals is positive. He omits the proofs of multiplication axioms because of the similarity to step 4.

(7) **Step 7** In this step, Rudin proves that $a0^* = 0^*a = 0^*$ by looking at different cases and using the axioms defined in step 6 to evaluate them. At the end of this step, Rudin concludes that $\mathbb{R}$ is an ordered field with the least-upper-bound property.

(8) **Step 8** In this next step, Rudin is getting very close to the idea that $\mathbb{R}$ can be constructed from $\mathbb{Q}$. For every $r \in \mathbb{Q}$, he associates it with the set $r^* \in R$ that satisfies three relations:
   (a) $r^* + s^* = (r + s)^*$
   (b) $r^*s^* = (rs)^*$
   (c) $r^* < s^*$ if and only if $r < s$

   By proving that these three are true, he is getting towards the conclusion.

(9) **Step 9** Based on step 8, Rudin showed that we can map a rational number r with its rational cut $r^* \in \mathbb{R}$ and that preserves addition, multiplication, and order. From this he concludes that $\mathbb{Q}$ is isomorphic to the ordered field $\mathbb{Q}^*$ whose elements are rational cuts.

1.2. What is the significance of showing that the real numbers can be constructed from the rationals? In other words, why did Rudin spend so much time proving this in the appendix?

The significance of showing that the real numbers can be constructed from the rationals and the reason Rudin spends so much time proving this is because the real numbers seem like a big blob of every single number that isn't imaginary. All of the other number systems such as the integers or the natural numbers feel like they have so much order, like the integers only being 0, positive, and negative whole numbers or the natural numbers only being positive whole numbers starting at 1. The real numbers feel much more random. By constructing the real numbers from the rationals, it gives order and understanding to a systemt that otherwise feels confusing. It also very nicely defines the irrationals, which seem very arbitrary to begin with.

1.3. What is your opinion of the writing of the proofs of the nine steps of Rudin's book?

To my surprise, I actually enjoyed Rudin's proof of the construction of $\mathbb{R}$ from $\mathbb{Q}$. I really like how he structured the entire proof as 9 different steps. It was slightly difficult trying to understand the proof as a whole when looking at individual parts. As far as the individual steps go, I think the proofs could have benefited from some more explanation of why we are doing what we are doing rather than saying this is what we want, writing a bunch of symbols, and concluding with this is what we got.

2. Problem 2

The goal of this problem is to fully understand the proofs of the root and ratio tests.

2.1. Read the proof of the Root Test (Theorem 3.33 on page 65). Do you feel the proof is clear?

Theorem 1 (Root Test). *Given $\sum a_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then, (a) if $\alpha < 1$, $\sum a_n$ converges; (b) if $\alpha > 1$, $\sum a_n$ diverges; (c) if $\alpha = 1$, the test gives no information.*

I think the part of the proof for the case where $\alpha < 1$ is pretty clear – since we already have familiarity with Theorem 3.17(b). However, I don't think the second case is extremely clear.

2.2. Critique the proof. Be sure to include at least one positive and one negative comment.

I think this proof nicely separates the cases and covers all the possibilities that can arise when evaluating series using the root test. One thing I like about the proof is it is very elegant which is nice for a more experienced reader but complicated for beginners. One thing I think it lacks is explanation of what it is doing and also forcing the reader to have to go to other parts of the book to check definitions that it cites.

2.3. New Version of the Proof

Root Test.

(1) Case 1: $\alpha < 1$. We know by the $\epsilon - N$ definition of an infinite limit that α is between a small epsilon window, $(\epsilon - \alpha, \epsilon + \alpha)$. We also know that this is less than 1 by assumption. We also know that $\forall n > N$, that $\sqrt[n]{|a_n|}$ will be in our epsilon window, which is less than 1 (Definition 3.17(b)). Let $0 < \beta < 1$. Then we can say $0 < \sqrt[n]{|a_n|} < \beta$ for some $n > N$. We can write this is as $|a_n|^{1/n} < \beta$, (we know this is positive because of the absolute value). Raising this inequality to the nth power: $|a_n| < \beta^n$. Applying the comparison test for convergence, we know that β^n converges by the geometric series test since $\beta < 1$. This implies that a_n also converges. Therefore $\sum a_n$ converges.

(2) Case 2: We assume that $\alpha > 1$, which means given $\epsilon > 0$ that $\forall n > N$, there is a sequence $\{n_k\}$ such that $\sqrt[n_k]{|a_{n_k}|}$ converges to α. By assumption, $\alpha > 1$, which means that the series $\sum a_n$ diverges because in order for a series to converge, the sequence needs to converge to 0. Therefore $\sum a_n$ diverges.

(3) Case 3: In order to show, that $\alpha = 1$ does not provide information about convergence or divergence, it is sufficient to find two series where the root test yields $\alpha = 1$ where one converges and the other diverges. Consider the series: $\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$. For both of these series, $\alpha = 1$, however $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges. Therefore $\alpha = 1$ is inconclusive.

Remark 1. *The changes I made to Rudin's proof include adding explicit definitions instead of sending the reader to a different part of the book, being more explicit about the geometric series test for case 1, and adding more explanation about what I am doing in each step to make the proof more reader friendly and less cryptic.*

□

3. Problem 3

Theorem 2 (Ratio Test)**.** *The series* $\sum a_n$

(a) *Converges if* $\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$,

(b) *Diverges if* $|\frac{a_{n+1}}{a_n}| \geq 1$ *for all* $n \geq n_0$, *where* n_0 *is some fixed integer.*

3.1. Read the proof of the Ratio Test (Theorem 3.34 on page 66). Do you feel the proof is clear?

I don't feel the proof is very clear. For example, the choosing of β seems arbitrary and case b is not really explained. I think the proof doesn't adequately explain the flow of logic and doesn't connect subsequent ideas, instead uses mathematical quantifiers which can lead to the proof feeling more confusing.

3.2. Critique the proof. Be sure to include at least one positive and one negative comment.

One thing I liked about the proof was it acknowledges the case of what happens when the limit is equal to 1, which was something I was wondering about after reading the previous proof. One thing I don't like about the proof is the lack of definitions and explanation of steps. I feel like older proofs especially tend to explain less of what they are doing, which is a good test for the reader to really understand, but it can also be quite frustrating when one gets stuck reading the proof. I think something that would've improved this proof was to say how β was chosen using the definition of the limit and the assumptions made in case (a).

3.3. Write a new version of the proof.

Ratio Test.

(1) Case 1: $\lim|\frac{a_{n+1}}{a_n}| = L < 1$. This means given $\epsilon > 0$, there exists N such that $\forall n > N$, $|\frac{a_{n+1}}{a_n}| < L+\epsilon < 1$. Denote $\beta = L+\epsilon$. Then we can write the inequality as $|\frac{a_{n+1}}{a_n}| < \beta$. Rearranging the inequality gives us: $|a_{N+1}| < \beta|a_N|$, for sufficiently large N. Since we know this inequality holds for any $n > N$, we can iterate by 1: $|a_{N+2}| < \beta|a_{N+1}|$ and we know that this is less than $\beta^2|a_N|$ because of the previous step. Repeating this process gives us:

$$|a_{N+p}| < \beta^p|a_N|.$$

This means that $\sum|a_{N+p}| < \sum\beta^p|a_N|$. Based on our earlier definition, we know that $\beta = L + \epsilon < 1$. Since it is less than 1, we know that the series on the right side converges by the geometric series test, which means that the left side series also converges by the comparison test. Since $\sum|a_{N+p}|$ converges, that implies $\sum a_n$ converges because for a series to converge, we are only focused on the tail end.

(2) Case 2: Let $\beta = L - \epsilon > 1$. This means that $\beta|a_n| < |a_{n+1}|$, which means that $|a_N| < |a_{N+1}|$, for all $n > N$. This means that the sequence of terms in a_n cannot go to zero since they are increasing and positive. For a series to converge, the sequence of terms must go to zero, which means that this series diverges. Therefore $\sum a_n$ diverges.

$\square$