# PRINCIPLES OF FINITE INDUCTION AND BINOMIAL THEORY

RYAN HAUSNER

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## 1. Problem 1

The first section of David Burton's *Elementary Number Theory*, often used as the text for Math 430 at USC, was a very enjoyable read. One of the things I appreciate about this text overall is how Dr. Burton goes through the proofs of each theorem instead of just stating the theorem. By seeing the proof of each theorem, it allows a basis to build intuition about the theory, which can then be applied to the problems. I also appreciate how he goes through several different exercises and uses a lot of inclusive pronouns such as "we" to make it seem like he is going through the problems with us rather than just stating information.

One of the parallels of the text to our class is the general structure. Similar to class, he starts with a general background, followed by a theorem, a proof of a theorem, then applications. Some parallels to the concepts we have covered in class so far are the use of the Well-Ordering Principle, proof by contradiction, and testing out different concepts using finite examples. The way he approaches some of the examples involving induction is very similar to how we approach problems in class, where we mess around with real examples of the theorems before writing our proof. For example, when demonstrating that induction doesn't really help when trying to formulate statements, he started with the list of equalities: $1 = 1$, $1 + 2 = 3$, $\ldots$, $1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 = 63$. This then helped him come to the statement: $1 + 2 + 2^2 + \ldots 2^{n-1} = 2^n - 1$. While we haven't spent much time deriving formulas in class, the idea of messing around with finite examples has been a staple in our proof writing experience.

As far as the content of the section goes, I enjoyed seeing the proof of the Archimedean Property as well as the proof of finite induction (which was almost identical to the proof we did in class except for some of the nomenclature). The examples of finite induction were very relevant and detailed, starting from easier examples to more difficult ones.

### 1.1. First and Second Principles of Finite Induction.

Theorem 1-2 (Principle of Finite Induction) as stated by Dr. Burton:

**Theorem 1.2.** *Let $S$ be a set of positive integers with the properties:*

(1) *1 belongs to S, and*
(2) *whenever the integer $k$ is in S, then the next integer k+1 must also be in S*

*Then S is the set of all positive integers.*

He calls Theorem 1-2, the "Principle of Finite Induction". While this is the Principle of Finite Induction, it can be divided into the First Principle of Finite Induction and the Second Principle of Finite Induction. To demonstrate the difference between the First and Second Principles of Induction, Dr. Burton doesn't explicitly define the First Principle, rather shows the difference needed when using the second. The main difference in the Second Principle of Finite Induction is in the second part of Theorem 1-2. He redefines it as 2', stating that "If k is a positive integer such 1,2,...,k belongs to S, then k+1 must also be in S. This means that the Second Principle of Induction takes into account all the values between $1, \ldots n$, where $n = k$ is the induction hypothesis. In terms of the ladder analogy used previously, it takes into account each step from 1 to $n$ and then follows that if $k + 1$ is in S, then the statement holds.

The First Principle of Finite Induction according to Dr. Burton is reflected in Theorem 1-2. This means that the First Principle of Induction is Theorem 1-2, where rather than looking at all the values from 1 to $n$, it starts at an $n$ value, where $n = k$ is the hypothesis, then shows that if $k+1$ is in $S$, then the statement holds. The First Principle of Finite Induction does not look at the positive integers before $n$ and starts at $n$ and goes from there. To demonstrate this difference, Dr. Burton uses the *Lucas Sequence*: $1, 3, 4, 7, \ldots$ where $a_n = a_{n-1} + a_{n-2}$. To prove the inequality: $a_n < (7/4)^n$, he uses the Second Principle of Finite Induction where proof is dependent on the successive values of $n$.

1.2. **Selected Exercise from 1.1.**

If $r \neq 1$, show that

$$a + ar + ar^2 + \ldots + ar^n = \tfrac{a(r^{n+1}-1)}{r-1}$$

*Proof.* This statement can be proven using the First Principle of Finite Induction (Theorem 1-2). Beginning with the base case, where $n = 1$, the left side of the equation is $a + ar$. The right side of the equation becomes

$\dfrac{a(r^2-1)}{r-1}$ where the numerator can be factored: $\dfrac{a(r-1)(r+1)}{r-1} = a(r+1) = ar+a$

, therefore the base case holds at $n = 1$. Our induction hypothesis is that for some $k \in S$,

$$a + ar + ar^2 + \ldots + ar^k = \frac{a(r^{k+1}-1)}{r-1}$$

. For the induction step, we assume the induction hypothesis and propose that the statement for $k+1$:

$$a + ar + ar^2 + \ldots + ar^k + ar^{k+1} = \frac{a(r^{k+2}-1)}{r-1}$$

Notice that in the induction step, $a + ar + ar^2 + \ldots + ar^k$ is included in the above statement. Since this is our induction hypothesis which we assume to be true, we can replace these terms with $\frac{a(r^{k+1}-1)}{r-1}$. We can now rewrite this as:

$$\frac{a(r^{k+1}-1)}{r-1} + ar^{k+1} = \frac{a(r^{k+2}-1)}{r-1}$$

. By algebraic manipulation, we can write this as

$$\frac{a(r^{k+1}-1) + ar^{k+1}(r-1)}{r-1} = \frac{a(r^{k+2}-1)}{r-1}$$

By distributing $a$, we get

$$\frac{ar^{k+1} - a + ar^{k+2} - ar^{k+1}}{r-1} = \frac{a(r^{k+2}-1)}{r-1}$$

Which simplifies to

$$\frac{a(r^{k+2}-1)}{r-1} = \frac{a(r^{k+2}-1)}{r-1}$$

Since the statement holds, we have proven by Induction (Theorem 1-2) that

$$a + ar + ar^2 + \ldots + ar^n = \frac{a(r^{n+1}-1)}{r-1}$$

holds for all $n$ if $r \neq 1$. □

## 2. Problem 2

Before delving into the selected exercise from section 1.2 of Burton's *Elementary Number Theory*, consider the following definitions and theorems from "1.2 The Binomial Theorem"[1]:

**Definition 1.** The binomial coefficients $\binom{n}{k}$, for any positive integer $n$ and any integer $k$ satisfying $0 \leq k \leq n$, defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

**Definition 2.** If $k = 0$ or $k = n$, then $\binom{n}{0} = \binom{n}{n} = 1$

**Theorem 2.1** (Pascal's Rule)**.**

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

### 2.1. Selected Exercise from 1.2.

For $n \geq 2$, prove that

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{n}{2} = \binom{n+1}{3}$$

*Proof.* Consider the base case, when $n = 2$, on the left side we have $\binom{2}{2}$. Using definition 2, we can compute the value of this: $\binom{2}{2}$ We know from Definition 2 that $\binom{2}{2} = 1$. Similarly, the right side becomes $\binom{2+1}{3}$. Using the same definition, we get that $\binom{3}{3} = 1$. Therefore the base case holds. For the inductive hypothesis, we assume that the statement holds for $k = n$, $k \geq 2$:

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{k}{2} = \binom{k+1}{3}$$

For the inductive step, we assume that the inductive hypothesis is true, and are trying to show that it holds for $k + 1$.

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{k}{2} + \binom{k+1}{2} = \binom{k+2}{3}$$

From the induction hypothesis that we hold to be true, we know that the sum of terms up to and including $\binom{k}{2}$ is equal to $\binom{k+1}{3}$. We can now rewrite our statement as

$$\binom{k+1}{3} + \binom{k+1}{2} = \binom{k+2}{3}$$

We can now apply Pascal's Rule (Theorem 2.1).[2]

$$\binom{u}{v} + \binom{u}{v-1} = \binom{u+1}{v}$$

Let $u = k + 1$, $v = 3$. Substituting these values into Pascal's Rule, we get:

$$\binom{k+1}{3} + \binom{k+1}{2} = \binom{k+2}{3}$$

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[1]The numbering of the definitions is different because Burton's text doesn't enumerate them.

[2]Since $n, k$ is being used as a part of the induction proof, we will use Pascal's Rule with $u, v$.

, which proves that this statement holds for $k+1$. Therefore by induction,

$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{n}{2}=\binom{n+1}{3}$$

is true for $n \geq 2$. $\square$