# Phase Space and Model Predictive Control

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# Contents

# 1 Part 1: Introduction to Phase Space

(a) The default on the website is $\frac{dx}{dt} = -x$ with the initial condition $(0, 1)$. Solve this differential equation explicitly.

$$\frac{dx}{x} = -dt$$

$$\int \frac{1}{x}\, dx = \int -1\, dt$$

$$\ln|x| = -t + C$$

$$e^{\ln|x|} = Ce^{-t}$$

$$1 = Ce^{0}$$
$$C = 1$$

$$x(t) = \boxed{e^{-t}} \tag{1}$$

(b) Explain how your answer in (a) matches with the vector field. Hint: What do the little dashed lines (vectors) represent?

The solution, $x(t) = e^{-t}$, from part (a) corresponds to the vector field for $\frac{dx}{dt} = -x$. The arrows in the vector field map the family of solutions for the differential equation. Each arrow represents a nudge in a certain direction based on the derivative that gives intuition into a solution to the differential equation.

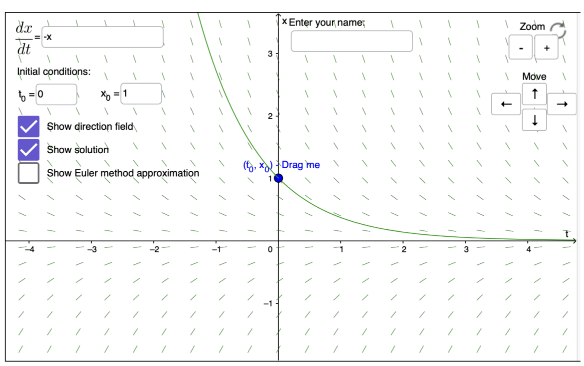


Figure 1: Vector field for $\frac{dx}{dt} = -x$.

As seen in Fig 1, each dashed line represents the slope of the solution curve at a particular point. Using the specific initial condition, we can see how vector fields map the solutions to not only a family of solutions but also a particular solution.

(c) Describe what a phase space is in general in the context of ODEs.

In the general context of ODEs, a phase space is a way of describing all possible states of a system, accounting for the dimensionality that differential equations bring based on the amount of variables. Phase space is an incredibly powerful tool because it is able to model multiple variables in an equation and using the stable states of the system it shows how a small change in one aspect will change the other aspects of the model. Phase space treats all states as a point in space, allowing for a visual interpretation of complex equations.

(d) Write down an ODE (i.e. Newton's law of cooling) and briefly describe its significance and meaning. If you are using a differential equation not from the textbook or course notes, include a citation of where you found it.

One of the most interesting applications of differential equations I found is a model that maps Romeo and Juliet's love for each other at a given time $t$.

$$\frac{dr}{dt} = a_{11}r + a_{12}j \tag{2}$$
$$\frac{dj}{dt} = a_{21}r + a_{22}j \tag{3}$$

[1]

Each parameter $a_{ik}$ can change the way their love responds to the other. By making $a_{11}$ and $a_{12} > 0$, it creates an "eager behavior" as described by Strogatz in his original paper.

(e) If your ODE has parameters, choose some values for those and enter your ODE on the website, include two screenshots of two different solutions for two different initial conditions. Alternatively, you can show two different vector fields by choosing two different values for your parameter(s).

The first set of parameters I chose for this system in matrix form is:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
$$\frac{dr}{dt} = a + j$$
$$\frac{dj}{dt} = -a + j$$

Since this set of differential equations is a system, I used Matlab to create the vector field.

For the next vector field, I changed the parameters using:

$$\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$$

In this case, both $r$ terms are negative and the $j$ term is doubled. The following vector field is the result:

[1]Steven Strogatz, *Love Affairs and Differential Equations*, Journal of Applied Mathematics, vol. 61, no. 1, pp. 123-135, 1988.

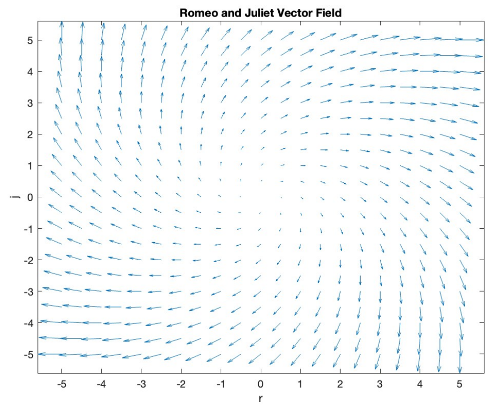


Figure 2: Vector field with parameters 1, 1, −1, 1

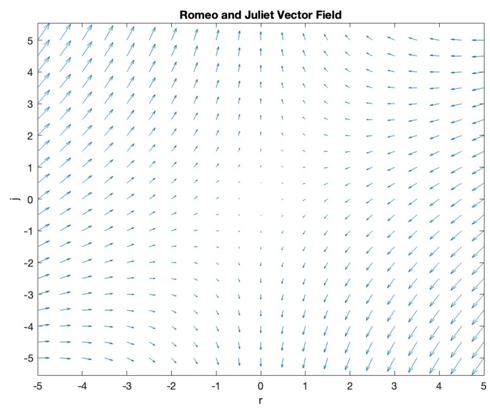


Figure 3: Vector field with parameters −1, 2, −1, 1

(f) What do you notice about the vector field on the website? Explain the context of what your ODE represents/models. For example, does your ODE have interesting steady states?

These vector fields offer tremendous insight into the system of differential equations describing Romeo and Juliet's love at a specific time. The first equation with the first set of parameters $\frac{dr}{dt} = r + j$ can be interpreted as Romeo's love for Juliet being influenced by both himself and Juliet. When both are acting in a positive direction (they share feelings for each other) the arrows in the vector field (phase space) are at a greater magnitude than when they are not acting in the same direction. This shows how something as abstract as phase space and vector fields can be applied to something so tangible (or not) as human feelings towards each other. For Juliet's equation with the first set of parameters $\frac{dj}{dt} = -r + j$, Juliet's own feelings contribute positively towards her feelings towards Romeo. However, Romeo's interaction contributes negatively towards her feelings for him. When he is more present, her feelings decrease but when he is more distant her feelings increase. This is reflected again in the phase space by the arrows of larger magnitude. When Romeo is very negative (distant) and Juliet is very positive, the arrows have a larger magnitude and a steeper slope as seen in the upper left quadrant of Figure 2.

For the other vector field with adjusted parameters, both equations involve Romeo contributing negatively towards the overall feelings. This can be interpreted as Romeo overthinking and Juliet favoring his distance. Again, the upper left quadrant has larger arrows with steeper slopes. For Juliet, Romeo again contributes negatively and herslef positively but of a smaller magnitude, as reflected in the upper left of Figure 3. This shows the power of phase space and how it can give insight into the dimensions of complex situtations in a way that can be interpreted both geometrically and intuitively.

This system has steady states when $(r, j)$ are both 0. This is found by setting $\frac{dr}{dt}$ and $\frac{dj}{dt} = 0$. The time when this system isn't changing is when both Romeo and Juliet are not having feelings about each other.

These vector fields also show how particular solutions are mapped based on their initial conditions. Using Matlab, I plotted two different particular solutions based on the first set of parameters.

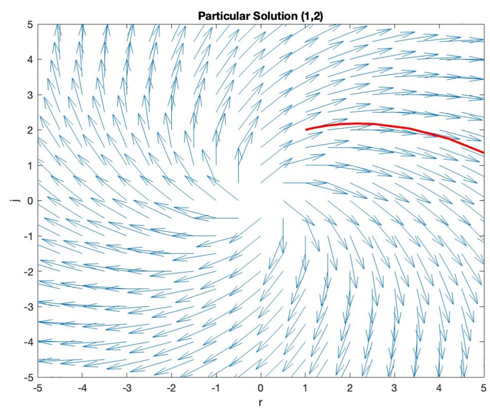


Figure 4: Particular Solution (1,2)

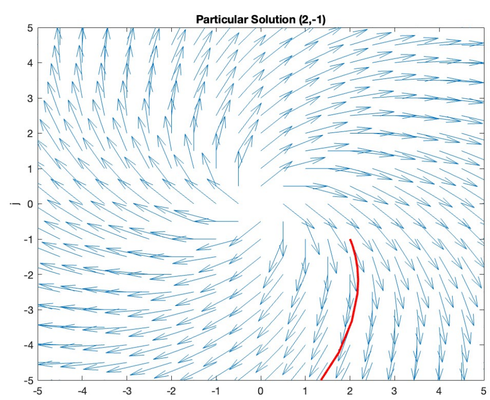


Figure 5: Particular Solution (2,-1)

As seen in Figures 4 and 5, the vector field is a way to visualize a differential equation (or system) without being bound by one particular solution. The particular solution is an infinitesmal way of finding a solution based on an initial value.

## 2 Part 2: Self Driving Cars

(a) Describe the equations used in the kinematic bicycle model. What are some variables the system uses and what are some assumptions made to make the system simpler?

The video goes through a variety of models to derive kinemtatic equations such as a point mass model, "stick" rigid body, and finally the kinematic bicycle model. The three equations used for the bicycle model are:

$$\dot{x} = vcos(\Psi + \beta) \tag{4}$$
$$\dot{y} = vsin(\Psi + \beta) \tag{5}$$
$$\dot{\Psi} = vcos(\beta)tan(\delta)/L \tag{6}$$

Equation 6 describes the change in the x-direction. It is directly proportional to the velocity. The $\cos(\Psi + \beta)$ term means that when the vehicle is pointed towards the y-axis, it is moving less in the x-direction. Equation 7 is very similar to equation 6 but in the y-direction. The same principles apply: it is directly proportional to velocity and when the vehicle is moving in the x-direction, it is moving less towards the y-direction. Equation 8 has to do with the steering angle. It has a steady state that when the turn angle is 0, the vehicle doesn't change direction.

The variables that are used to describe this model are:

- $O$ = Instantaneous Center of Rotation
- $\delta$ = Steering Angle
- $R$ = Radius of Curvature; $\bar{R}$ = Radius of Curvature for Rear Axle
- $\Psi$ = Heading Angle
- $V$ = Velocity for Center of mass
- $\beta$ = Body Slip Angle
- $L_f$ = Distance from Front Axle to Center of Mass; $L_r$ = Distance from Rear Axle to Center of Mass

The assumptions that are made to make this system simpler are that the angle $\beta$ isn't influenced by other factors as it is in the real world, there exists an instantaneous center of rotation, and there is no slipping of the wheel.

(b) Describe model predictive control and its role in autonomous vehicles. How does it relate to the kinematic bicycle model?

Model predictive control is a method used for making predicitons about how to make future decisions using the current situation and contrainsts given. For example, with self-driving cars, the current position, velocity, and acceleration of the vehicle can be used in addition to the future layout of the road to determine the velocity and turn angle to make in order to stay on the road. The constraints that could be used are speed limit, the minimum distance between cars, and physical contraints such as acceleration time to reach a certain velocity.

(c) We will now focus on the simplified kinematic bicycle model. Consider the kinematic bicycle equations outlined in the second video along with some assumptions:

$$\frac{dx}{dt} = v(t)cos(\Psi(t) + \beta) \tag{7}$$
$$\frac{dy}{dt} = v(t)sin(\Psi(t) + \beta) \tag{8}$$
$$\frac{d\Psi}{dt} = v(t)cos(\beta)tan(\delta)/L \tag{9}$$
$$\beta = \tan^{-1}(\frac{L_R}{L}tan(\delta)) \tag{10}$$

The following assumptions are made:

- The center of mass of bicycles is slightly closer to the rear wheel than the front. We take $\frac{L_R}{L} = 0.65$.
- Assume the distance between the wheels is $L = 1$.
- We'll assume the cyclist can accelerate at a constant $a = 0.25m/s^2$ making $v(t) = at = 0.25t$.

*i)* Rewrite the system using these assumptions.

$$\begin{aligned}
\frac{dx}{dt} &= 0.25t\cos(\Psi(t) + \beta), \\
\frac{dy}{dt} &= 0.25t\sin(\Psi(t) + \beta), \\
\frac{d\Psi}{dt} &= 0.25t\cos(\beta)\tan(\delta), \\
\beta &= \tan^{-1}(0.65\tan(\delta)).
\end{aligned}$$

*ii)* Choose a turning angle $0 < \delta < \pi/2$ and, using technology, find the value of $\beta$.
For $\delta = \pi/4$:

$$\begin{aligned}
\beta &= \tan^{-1}(0.65\tan(\pi/4)), \\
\beta &= \tan^{-1}(0.65), \\
\beta &= 0.57638.
\end{aligned}$$

*iii)* Solve the differential equation $\frac{d\Psi}{dt} = v(t)\cos(\beta)\tan(\delta)/L$ with $\Psi(0) = 0$.

$$\begin{aligned}
\frac{d\Psi}{dt} &= 0.25t\cos(0.57638)\tan(\pi/4), \\
\Psi(t) &= \int 0.2095t\,dt, \quad \Psi(0) = 0, \\
\Psi(t) &= 0.1048t^2 + C, \\
\Psi(t) &= \boxed{0.1048t^2}
\end{aligned}$$

*iv)* Solve the first two differential equations $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 0.25t\cos(0.1048t^2 + 0.57638),$$
$$x(t) = \int 0.25t\cos(0.1048t^2 + 0.57638)\,dt$$
$$\text{Let } u = 0.1048t^2 + 0.57638, \quad \frac{du}{0.2096t} = dt,$$
$$x(t) = \int 1.19269\cos(u)\,du$$
$$x(t) = 1.19269sin(0.1048t^2 + 0.57638) + C \qquad \text{Using initial condition } x(0) = 3$$
$$x(t) = \boxed{1.19269sin(0.1048t^2 + 0.57638) + 2.35}$$

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$$\frac{dy}{dt} = 0.25t\sin(0.1048t^2 + 0.57638),$$
$$y(t) = \int 0.25t\sin(0.1048t^2 + 0.57638)\,dt$$
$$\text{Let } u = 0.1048t^2 + 0.57638, \quad \frac{du}{0.2096t} = dt,$$
$$y(t) = \int 1.19269\sin(u)\,du$$
$$y(t) = -1.19269cos(0.1048t^2 + 0.57638) + C \qquad \text{Using initial condition } y(0) = 4$$
$$y(t) = \boxed{-1.19269cos(0.1048t^2 + 0.57638) + 5}$$

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*v)* Using technology, graph the parameterized curve $(x(t), y(t))$ Does the graph make sense? Describe how your answer relates to the initial conditions you choose.

$$\boxed{(x(t), y(t)) = (1.19269sin(0.1048t^2 + 0.57638) + 2.35,\ -1.19269cos(0.1048t^2 + 0.57638) + 5)} \tag{11}$$

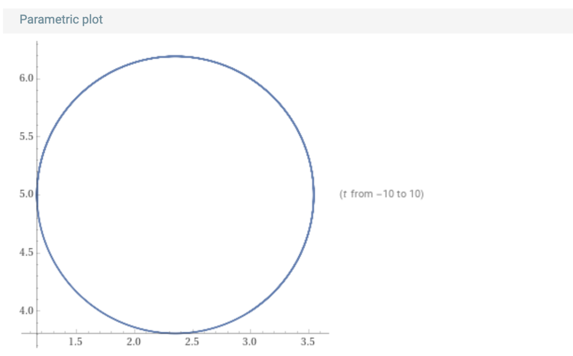


Figure 6: Parametric Graph (Eq. 13)

This graph does make sense due to the constant turn angle. Since the bicycle is turning at a constant angle, it makes sense it would form a circle and end up at the point it started it based on the initial conditions $x(0) = 3$, $y(0) = 4$. The answer relates to the initial conditions because the angle chosen ($\pi/4$) is constant which explains the circular path. The path of the bicycle also makes sense because it starts at $(3, 4)$ and gets further away until it returns.

*vi)* Describe how the path and velocity of the bicycle are changing as time increases. Do you think the model is realistic?

As time increases, the path forms a curved pattern and ends at the starting point, forming a circle. The velocity ($0.25t$) is linear but since there is a circular path, it can be interpreted as the bicycle going around the same path but at a faster speed. Due to the circular graph and the increasing velocity, the fact that it forms a circle coupled with the constant turn angle implies that the bicycle is making a continuous loop but at faster speeds.

This model is not very realistic because it doesn't account for physical factors such as friction and other physical constraints such as ability to accelerate. It is also limiting because it only allows a constant turn angle. For an actual model predictive control, varying angle would be necessary in order to make predictions based on surroundings, current trajectory, and constraints. Despite the shortcomings, the bicycle model still provides a framework for modeling with differential equations and how it can be used to predict future states based on current factors.

# A Code Segments

```
f_r = @(r, j) r + j;
f_j = @(r, j) -r + j;

[r, j] = meshgrid(-5:0.5:5, -5:0.5:5);

dr = f_r(r, j);
dj = f_j(r, j);

figure;
quiver(r, j, dr, dj);
xlabel('r');
ylabel('j');
title('Romeo and Juliet Vector Field');
axis tight;
```

Listing 1: Matlab Code for Vector Fields