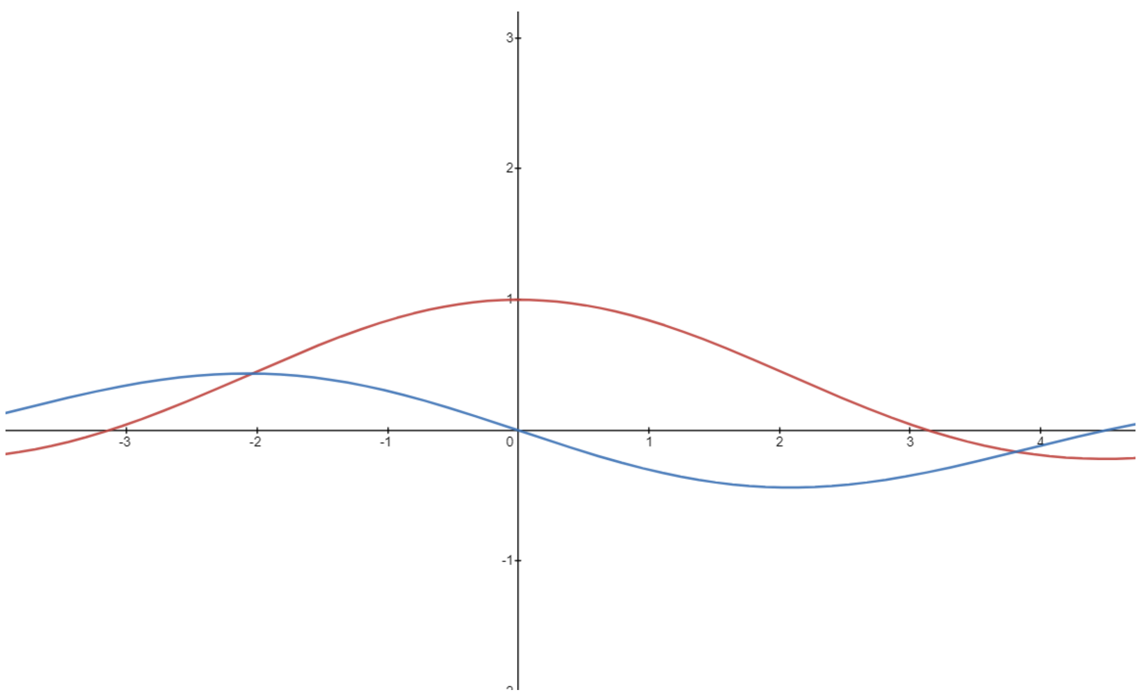
INTEGRATION OF NON-ELEMENTARY ANTIDERIVATIVES

Ryan Hausner



Introduction and Background

This paper is an informative text on one solution to the non-elementary antiderivatives of the Gaussian integral and the Sin integral. The Gaussian integral (also known as the error function) is e^{-x^2}, the Sin integral (also known as the Sinc function) is $\frac{\sin(x)}{x}$, and the Bernoulli Integral is x^{-x}. These functions have various applications in areas such as statistics and electrical engineering.

Elementary Antiderivatives

Elementary functions are polynomials, rational functions, power functions, exponential functions, logarithmic functions, trigonometric functions, and inverse trigonometric functions. These can all have an antiderivative that can be evaluated using normal means of calculus.

Example 1.

$$\int x^2 dx = \frac{1}{3}x^3 + c$$

While this is a very simple example, it demonstrates the antiderivative, or the integral of a very simple function. Examples 1 and 2 (shown below) are examples of elementary integrals.

Example 2.

$$\int e^{2x} dx \qquad u = 2x \qquad \frac{du}{2} = dx$$

$$\frac{1}{2}\int e^{u} du = \frac{e^{2x}}{2} + c$$

Non-Elementary Antiderivatives

While most functions have an elementary antiderivative, not all functions do. Those functions who don't are called non-elementary functions. This means that when evaluating the antiderivative, normal means of calculus can't be used to fully solve them.

Improper Integrals

In simple terms, an integral is improper if its upper bound, lower bound, or both are infinite.

Example 3.

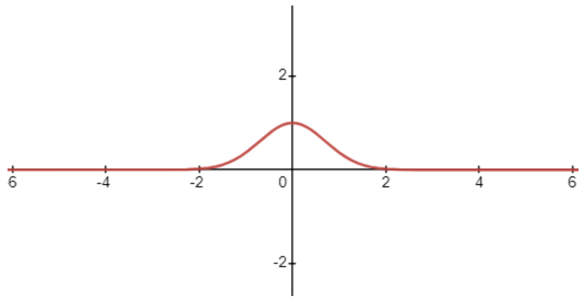
$$\int_0^{\infty} f(x)dx = \lim_{a\to\infty}\int_0^{a} f(x)dx$$

Using a limit as “a” approaches infinity is the proper way to express an improper integral.

Evaluation of the Gaussian Integral

The Gaussian function, $I(x) = e^{-x^2}$ is a non-elementary function; this means normal means of evaluating integrals will not suffice. This integral is evaluated from -∞ to ∞.

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$



At first glance, you might be tempted to use u-substitution. But that will not work here. Since this is non-elementary, other means must be used.

Since this integral is equal to “I”, we can square “I” and make it two integrals.

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$$

Rewrite as:

$$I^2 = \iint_{-\infty}^{\infty} e^{-x^2} \cdot e^{-y^2} dydx$$

$$I^2 = \iint_{-\infty}^{\infty} e^{-(x^2+y^2)} dydx$$

Replace $x^2 + y^2 = r^2$

When converting from cartesian to polar coordinates, the Jacobian r is introduced into the integral and the bounds of integration change.

$$I^2 = \int_{0}^{\infty} \int_{0}^{2\pi} e^{-r^2} r \, drd\theta$$

$$\int_{-\infty}^{0} \int_{0}^{2\pi} e^{-r^2} r \, drd\theta$$

Evaluating the inner integral first, this becomes a single variable integral.

$$2\pi \int_{-\infty}^{0} e^{-r^2} r \, dr d\theta \qquad u = -r^2 \qquad \frac{du}{-2r} = dr$$

$$\pi \int_{-\infty}^{0} e^{u} \, du$$

Applying the Fundamental Theorem of Calculus:

$\pi[e^0 - e^{-\infty}] = \pi$

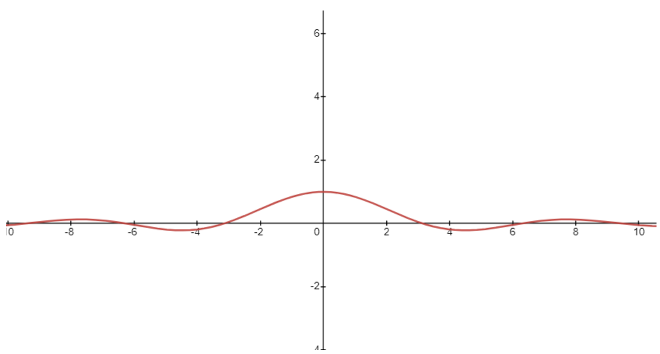
Recall that this is the solution for I^2, therefore, to get the solution for I, we square root both sides.

$I = \sqrt{\pi}$

Therefore, $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

Evaluation of the Sinc Function

The sinc function is defined as$f(x) = \frac{\sin(x)}{x}$.



An important concept to understand when evaluating integrals if whether the function is even or odd.

A function is even if $f(-x) = f(x)$.

If a function is even, $\int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx$.

$$\frac{\sin(-x)}{x} = \frac{-\sin(x)}{-x} = \frac{\sin(x)}{x}$$

Since this function is even, we can evaluate it from 0 to ∞.

We will be using Feynman's Technique to evaluate this integral. Feynman's Technique introduces another parameter to the function to make integrating possible. Feynman's Technique says we can find the area under the curve by taking the derivative of the integral.

$$\int_0^{\infty} \frac{\sin(x)}{x} dx$$

$I(a) = \int_0^{\infty} e^{-ax} \cdot \frac{\sin(x)}{x} \mathrm{dx}$

Now we differentiate.

$$I'(a) = \frac{d}{da} \int_0^{\infty} e^{-ax} \cdot \frac{\sin(x)}{x} dx$$

When taking the derivative with respect to "a" of an integral with respect to "x" we can take the partial derivative with respect to "a".

$\int_0^{\infty} \frac{\partial}{\partial a} e^{-ax} \cdot \frac{\sin(x)}{x} \mathrm{dx}$

When taking the partial derivative with respect to "a", x is treated as a constant.

$$I'(a) = \int_0^{\infty} -xe^{-ax} \cdot \frac{\sin(x)}{x} dx$$

This leaves us with an integral we can evaluate using Laplace Transforms.

Note, the Laplace transform for $e^{-at} \sin(bt)\, dt$ is $\frac{b}{(s+a)^2+b^2}$.

$$I'(a) = \int e^{-ax} \sin(x)\, dx$$

$$L\{I'(a)\} = \frac{-1}{1+a^2}$$

As shown, the Laplace Transform easily evaluates the integral and saves using integration by parts twice.

Now that we have $I'(a)$, we must integrate both sides.

$$\int I'(a) = -\int \frac{1}{1+a^2} da$$

$I(a) = -\tan^{-1}(a) + c$

Now we have to solve for c.

$I(0) = c$

$I(\infty) = -\tan^{-1}(\infty) + c$

$$c = \frac{\pi}{2}$$

$$I(a) = \tan^{-1}(a) + \frac{\pi}{2}$$

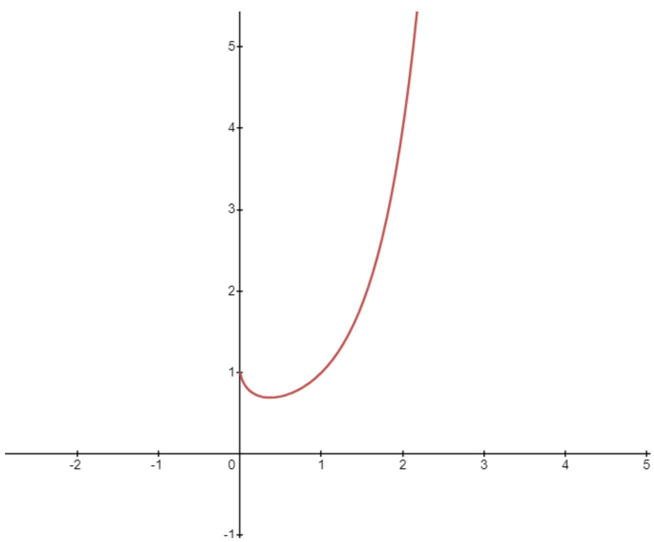
Therefore,

$$\int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = \pi$$

Evaluation of the Bernoulli Integral

The Bernoulli Integral is the third and final non-elementary integral we are going to evaluate. The Bernoulli Integral: $\int_0^1 x^{-x} dx$. This integral requires an even further understanding of topics in calculus such as infinite series and the Gamma Function.



The Gamma Function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

This function is also known as the factorial function.

$\Gamma(n) = (n-1)!$

Keeping this in mind, we can begin evaluating the integral.

$$\int_0^1 x^{-x} dx$$

$$\int_0^1 e^{\ln(x)^{-x}} dx$$

Using properties of logarithms:

$$\int_0^1 e^{-x ln(x)} dx$$

Recall the power series for e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Now we are going to substitute x for $-xln(x)$.

$$\int_0^1 \sum_{n=0}^{\infty} \frac{(-xln(x))^n}{n!} dx$$

Because of uniform convergence, we can swap the integral and the summation.

$$\sum_{n=0}^{\infty} \int_0^1 \frac{(-xln(x))^n}{n!} dx$$

Now we use u-substitution:

$u = -\ln(x) \qquad -u = \ln(x) \qquad e^{-u} = x$

$-e^{-u} du = dx$

Remember to also change the bounds of integration.

Upper Bound: $-\ln(1) = 0$

Lower Bound: $-\ln(0) = \infty$

$$\sum_{n=0}^{\infty} \int_{\infty}^{0} \frac{u^n e^{-nu}}{n!} \cdot (-e^{-u})\, du$$

Notice how we need to swap the upper and lower bounds of the integral. This can be done by negating the integral.

$$\sum_{n=0}^{\infty} \int_{0}^{\infty} \frac{u^n e^{-nu}}{n!} \cdot (e^{-u})\, du$$

$$\sum_{n=0}^{\infty} \int_{0}^{\infty} \frac{u^n e^{-(n+1)u}}{n!}\, du$$

Doing another u-substitution:

$v = (n+1)u \qquad dv = (n+1)du$

$$\sum_{n=0}^{\infty} \int_{0}^{\infty} \frac{v^n}{(n+1)^n} \cdot \frac{e^{-v}}{n!} \cdot \frac{dv}{n+1}$$

The first two terms with n in the denominator can be treated as constants so we can pull them outside of the integral.

$$\sum_{n=0}^{\infty} \frac{1}{n!\,(n+1)^{n+1}} \int_{0}^{\infty} v^n e^{-v} dv$$

Notice we have the Gamma Function, so we can replace the entire integral expression with $n!$

$$\sum_{n=0}^{\infty} \frac{1}{n!\,(n+1)^{n+1}} \cdot n!$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}}$$

If we start at $n = 1$ instead of $n = 0$, then we can change $n + 1$ to n.

$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

$$\sum_{n=1}^{\infty} n^{-n}$$

$$\int_{0}^{1} x^{-x}\, dx = \sum_{n=1}^{\infty} n^{-n}$$

When we evaluate this integral, we find that the answer is the same as the integral in the form of a summation.

Conclusion

Non-elementary antiderivatives are extraordinarily complex and require careful manipulation and a comprehensive understanding of calculus and further techniques such as Feynman's Technique and Laplace Transforms.

Proof 1.

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dy dx$$

$$= \int_{0}^{\infty} \int_{0}^{2\pi} e^{-R^2} R \, dR \, d\theta$$

$$= \int_{-\infty}^{0} \int_{0}^{2\pi} e^{-R^2} R \, dR \, d\theta \qquad u = R^2$$

$$du = -2R \, dR$$

$$\frac{du}{-2R} = dR$$

$$= 2\pi \int_{-\infty}^{0} e^{-R^2} R \, dR$$

$$= 2\pi \int_{-\infty}^{0} e^{u} R \cdot \frac{du}{2r}$$

$$= \pi \int_{-\infty}^{0} e^{u} du$$

$$= \pi \left[e^{0} - e^{-\infty} \right]$$

$$= \pi(1)$$

$$= \pi$$

$$I^2 = \pi$$

$$\boxed{I = \sqrt{\pi}}$$

Proof 2.

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx$$

$$f(-x) = \frac{\sin(-x)}{-x}$$

$$\frac{-\sin(x)}{-x}$$

$$\frac{\sin x}{x}$$

$$f(-x) = f(x)$$

$$\int_{0}^{\infty} \frac{\sin x}{x}\,dx$$

$$\int_{0}^{\infty} e^{-ax} \cdot \frac{\sin x}{x}\,dx$$

$$I(a) = \int_{0}^{\infty} e^{-ax} \cdot \frac{\sin x}{x}\,dx$$

$$I'(a) = \frac{d}{da}\int_{0}^{\infty} e^{-ax} \cdot \frac{\sin x}{x}\,dx$$

$$I'(a) = \int_{0}^{\infty} \frac{\partial}{\partial a} e^{-ax} \cdot \frac{\sin x}{x}\,dx$$

$$I'(a) = \int_{0}^{\infty} -x e^{-ax} \cdot \frac{\sin x}{x}\,dx$$

$$\int e^{-ax} \sin(x)\,dx$$

$$\int I'(a) = -\int \frac{1}{1+a^2}\,da$$

$$I(a) = -\tan^{-1}(a) + c$$

$$I(0) = -\tan^{-1}(0) + c$$

$$I(0) = c$$

$$I(\infty) = 0$$

$$I(\infty) = -\tan^{-1}(\infty) + c$$

$$c = \pi/2$$

$$I(a) = \tan^{-1}(a) + \pi/2$$

$$I(0) = \int_0^\infty \frac{\sin(x)}{x}\,dx$$

$$I(0) = \tan^{-1}(0) + \pi/2$$

$$I(0) = 0 + \pi/2$$

$$I(0) = \pi/2$$

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \pi/2$$

$$\int_{-\infty}^\infty \frac{\sin(x)}{x}\,dx = \pi$$

Proof 3.

$$\int e^{-ax} \sin(x)\, dx$$

$u = e^{-ax}$ $\quad dv = \sin(x)\, dx$

$du = -ae^{-ax} dx$ $\quad V = -\cos(x)$

$$-e^{-ax}\cos x - a\int e^{-ax}\cos(x)\, dx$$

$u = e^{-ax}$ $\quad dv = \cos(x)$

$du = -ae^{-ax} dx$ $\quad V = \sin(x)$

$$-e^{-ax}\cos(x) - a\left(e^{-ax}\sin(x) + a\int e^{-ax}\sin x\, dx\right)$$

$$-e^{-ax}\cos(x) - ae^{-ax}\sin x - a^2\int e^{-ax}\sin(x)\, dx$$

$$\int e^{-ax}\sin x\, dx = -e^{-ax}\cos x - ae^{-ax}\sin x - a^2\int e^{-ax}\sin x\, dx$$

$$\int e^{-ax}\sin x\, dx + a^2\int e^{-ax}\sin x\, dx = -e^{-ax}\cos x - ae^{-ax}\sin x$$

$$(1 + a^2)\int e^{-ax}\sin x\, dx = -e^{-ax}\cos x - ae^{-ax}\sin x$$

$$\int e^{-ax}\sin(x)\, dx = \frac{-e^{-ax}\cos x - ae^{-ax}\sin x}{1 + a^2}$$

$$I(a)\int_0^{\infty} e^{-ax}\sin x\, dx$$

$$\mathcal{L}\{I(c)\} = \frac{1}{1+c^2}$$

$$I'(c) = \frac{1}{1+c^2}$$

$$\int I'(c) = \int \frac{1}{1+a^2}$$

$$I(a) = +$$

Proof 4.

$$F(s) = \mathcal{L}\{f(t)\}$$

$$\mathcal{L}[e^{-tx^2}] = \int_0^\infty e^{-tx^2} e^{-st} \, dt$$

$$\int_0^\infty e^{-t(x^2+s)} \, dt$$

$u = -t(x^2+s)$

$\frac{du}{dt} = -(x^2+s)$

$dt = \frac{1}{-x^2-s} \, du$

$$\frac{1}{-x^2-s} \int_0^\infty e^u \, du$$

$$\frac{1}{-x^2-s} \left[e^\infty - e^0 \right]$$

$$= \frac{1}{x^2+s}$$

Proof 5.

$$\mathcal{L}\left[e^{\lambda t}\right] = \int_0^\infty e^{-st} e^{\lambda t}\, dt$$

$$\int_0^\infty e^{-t(s-\lambda)}\, dt$$

$$u = -t(s-\lambda)$$

$$-\frac{1}{(s-\lambda)} \int_0^\infty e^{-t(s-\lambda)}\, dt$$

$$\frac{du}{dt} = -(s-\lambda)$$

$$dt = \frac{1}{-(s-\lambda)}\, du$$

$$-\frac{1}{s-\lambda}\left[e^{\infty} - e^{0}\right]$$

$$\frac{-1}{s-\lambda}(-1)$$

$$= \frac{1}{s-\lambda}$$

Proof 6.

$$\int_0^1 x^{-x}\,dx$$

$$\int_0^1 e^{\ln x^{-x}}\,dx$$

$$\int_0^1 e^{-x\ln x}\,dx$$

$$e^x \approx \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\int_0^1 \sum_{n=0}^{\infty} \frac{(-x\ln x)^n}{n!}\,dx$$

$$\sum_{n=0}^{\infty} \int_0^1 \frac{(-\ln x)^n x^n}{n!}\,dx$$ Uniform convergence of this sum - swap integral + sum

$u = -\ln(x)$

$-u = \ln x$

$e^{-u} = x$ ← Change bounds $0 \to \infty$, $1 \to 0$

$-e^{-u}\,du = dx$

$$\sum_{n=0}^{\infty} \int_{\infty}^{0} \frac{u^n e^{-nu}}{n!}(-e^{-u})\,du$$

$$\sum_{n=0}^{\infty} \int_0^{\infty} \frac{u^n e^{-nu}}{n!}\cdot(e^{-u})\,du$$ swap order of bounds by negating it

$$\sum_{n=0}^{\infty} \int_0^{\infty} \frac{u^n e^{-(n+1)u}}{n!}\,du$$ $V = (n+1)u$, $dV = (n+1)\,du$

$$\sum_{n=0}^{\infty} \int_0^{\infty} \frac{V^n}{(n+1)^n}\frac{e^{-V}}{n!}\frac{dV}{n+1}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!\,(n+1)^{n+1}} \underbrace{\int_0^{\infty} V^n e^{-V}\,dV}_{\text{gamma function}}$$ $\Gamma(n+1) = n!$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^n}$$

$$= \sum_{n=1}^{\infty} n^{-n}$$