Music and Mathematics: Can music be quantified?

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Abstract

Two of the oldest and most beautiful disciplines throughout history are music and mathematics. While some may think these are two distinct fields, they intersect and create a deeper interpretation of both. The goal of this paper is to examine the union of both music and mathematics, to discover the deeper connection between the two and the insight that is uncovered when doing so. The mathematics of this paper will be kept brief in order to place an emphasis on the musical aspects. Several musical constructs such as pitch, intervals, harmonies, and overtones will be explored with mathematical context to gain insight into how the mathematics behind these musical components can help on a musical level. Additionally, the paper will delve into how music can act as a catalyst for learning mathematics and the converse.

Introduction

Music surrounds us. Whether it's blasting in our headphones or in the background at the supermarket, it is present in our lives. The non-axiomatic aspect of the music in our lives are the underlying principles that govern our universe: mathematics. The way mathematics allows us to describe our surroundings is slightly different than the way music does. While music has tangible ways of creating impact via instruments, mathematics does not. The mathematical instruments that describe music are concepts that can't be seen, heard, or felt; yet they provide valuable insight that is otherwise not presented. Musical compositions, ranging from ancient times to the newest albums, contain the same fundamental musical constructs that we often overlook. These include the nature of pitch itself, musical intervals, and overlays and harmonics.

Pitch as Frequency

What is sound? We hear sound in everything, but how often do we ask ourselves what is actually happening behind the scenes. Sound is the propagation of waves through the air caused by vibrations.[1] These waves that travel through the air reach our air and are filtered by frequency by a membrane and microscopic hairs in the ear and are then turned into signals that our brain interprets as sound.[2] While the concept of sound as invisible waves moving through the air and strangely entering our ear seems abstract, it offers a new way of understanding music. These waves oscillate at different frequencies depending on the type of vibration. These different frequencies are known as pitch, the first major intersection between music and mathematics. Each musical note has a wave representation that contains information about its frequency in the units Hertz (Hz). For example, an *A* note in standard tuning above middle *C* creates a wave that oscillates at 440 Hz.[3] This lays the groundwork for music and mathematics. By being able to describe musical notes as mathematical objects, we can now begin to understand music using the power and analysis that math provides.

Intervals

Pythagoras began to lay the foundation for the interdisciplinary crossover between the two subjects with his discovery of intervals as ratios. One of the most important ideas of music is the octave, which contains 8 subsequent notes. Through experimentation, Pythagoras was able to determine a mathematical ratio to describe this, 1:2.[4] While this may just be two numbers, it represents one of the most important ideas of music. This ratio means that if two notes are played where one is double the frequency of the other, then they form an octave. This allowed Pythagoras to use mathematics to explore different musical ideas such as a perfect fourth, perfect fifth with ratios 3:4 and 2:3 respectively. Furthermore, he was able to ascertain ideas such as

[1] George Audsley, *What is Sound? The Substantial Theory versus the Wave Theory of Acoustics Proceedings of the Musical Association, 16th Sess. (1889–1890), 105.*

[2]1. National Institute on Deafness and Other Communication Disorders, "How Do We Hear?" https://www.nidcd.nih.gov/health/how-do-we-hear#:~:text=Sound%20waves%20enter%20the%20outer,bones%20in%20the%20middle%20ear.

[3] David Wright, *Mathematics and Music* (Self-published), 4.

[4] Richard Crocker, *Pythagorean Mathematics and Music. The Journal of Aesthetics and Art Criticism* 22, no. 2 (1963): 189.

combining ratios to form other intervals. When combining 2:3 (fifth) and 3:4 (fourth), the result was 2:4 (an octave).[5] Discoveries that were made in 400 B.C,[6] still have an impact on music today that contains the same concepts. While Pythagoras was not a musician or composer, his discoveries of musical intervals as mathematical ratios allow the view of music through a different perspective, where notes are pitch and octaves are ratios. Intervals as ratios describe why playing one note and another note that is double the frequency sound so perfect together. The notion of pitch as frequency creates even further musical insight. Higher frequencies oscillate at a faster rate, resulting in waves that cause quicker air propagation. These higher frequencies are more direct in how they travel, compared to lower frequencies which are less directional and fill the space more.[7] By using this, composers can write parts accordingly and position different instruments in different locations based on their range of frequencies.

Harmonics and Overtones

One of the next most important ideas that falls in the cross sections of music and mathematics are harmonics and overtones. Harmonics are notes where their frequency is an exact multiple (meaning an integer scale) of the fundamental note.[8] An example of a harmonic is picking a note with frequency f and another note that is an integer number times that, for example $2f$. Other harmonics include $3f$, $4f$, $5f$, and so on. The most pleasing harmonics are those with the smallest ratio of numbers and often contain powers of *2*, *3*, or *5*.[9] Not only can mathematics describe musical phenomena, but it can provide insight into what types of frequencies are pleasing. The smaller ratios of smaller numbers are more pleasing than those of larger numbers. Additionally, harmonics containing a power of *2*, *3*, or *5* are more pleasing to the

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[5] Crocker, *Pythagorean Mathematics and Music*, 193.
[6] Crocker, *Pythagorean Mathematics and Music*, 190.
[7] Susan Wollenberg, *Music and Mathematics: An Overview (*Oxford: Oxford University Press, 2006), 47.
[8] F.J. Budden, *Modern Mathematics and Music*. T*he Mathematical Gazette* 51, no. 377 (1967): 206.
[9] Budden, *Modern Mathematics and Music*, 206

ear than those of other powers. Not only can harmonics be described as multiples of frequencies, but also overlapping waves. When playing a single note on the piano, it creates a single pitch, and thus a single wave that is propagating through the air. However, when multiple pitches are played at the same time, there are multiple waves moving through the air. Each of these can be described by the equation: $d_k sin(2\pi Fkt + \beta_k)$. The *sin* part of the equation describes the wave like motion, d_k is the amplitude or how loud the sound is, $2\pi F$ is the frequency, *t* is time, and β_k is a phase shift that moves the sound left or right (to describe different sounds being played after or before one another). This equation is a true encapsulation of the interplay between music and math, showing how each note that is bowed by a violinist or sung by a vocalist is contained by a mathematical equation. One of the most important variables in this equation is *k*, which represents the harmonics and overtones. When playing a note that has a pitch of *f*, the variable *k* shows how integer multiples of that frequency are harmonics. Picking *k=1* gives you the fundamental, which is the first harmonic. Increasing *k* by one gives you *2f* or the second harmonic (first overtone). The difference between harmonics and overtones is that harmonics start counting at the fundamental, the first note played, and overtones start at the first multiple of the frequency. The first harmonic is the k^th harmonic and the first overtone is denoted by (k-1).[10] The importance in differentiating between overtones and harmonics is that the first harmonic (the fundamental note) doesn't have as much of a listening impact as the first overtone which contributes to the timbre of the sound. Timbre is the acoustic quality that gives a sound its uniqueness, the way there is a noticeable difference between a trumpet playing and a violin playing.[11] Overtones enrich the principal note, allowing for a richer sound.[12] By using math, musicians can use a mathematical toolbox to decompose sound as well as optimize it for certain

[10] Wright, *Mathematics and Muisc*, 114.
[11] Stephen McAdams, *Musical Timbre Perception*. *The psychology of music* 3 (2013): 35.
[12] Frank Lawlis, *The Basis of Music-Mathematics*. *The Mathematics Teacher* 60, no. 6 (1967): 593.

conditions. These concepts create a foundation that allows musicians and composers to figure out what notes to play in a certain key or why some combinations of notes sound better than others.

Visualization of Topics

To create a more understanding of these abstract concepts, picture a beginner piano, where instead of the names of the notes on the keys, the frequencies of the pitch are instead labeled. The first note is a white key - middle C. As you move to the left, the frequency numbers get lower and as you move to the right, they get higher. If you find two notes where one is double the other (200 and 400 Hz for example), they form an octave. The same applies for all the other intervals as discovered by Pythagoras. When fiddling around on the piano, if one plays a note, then another note twice the frequency, then another note three times the frequency, and so on, this person is playing harmonics and overtones. These concepts are abstract in theory, but make sense in practice. By experimenting on a piano, the same discoveries that Pythagoras made can be replicated.

The Effects of Music on Math

Not only can mathematics enhance music, but music can also enhance mathematics, especially learning it. Several studies have shown that those who play instruments perform better in mathematics than those who do not. Researcher Kathryn Vaughn conducted a study to find a relationship between playing an instrument and learning mathematics and found a positive association between the voluntary study of music and mathematical achievement.[13] Music is full of math and math is expressed by music. The double sided connection between the two disciplines allows for deeper intuition into both. Further research has shown that students who learn instruments perform better in areas of geometry such as 2D/3D shapes and symmetric

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[13] Vaughn (page 154) Kathryn Vaughn, *Music and Mathematics: Modest Support for the Oft-Claimed Relationship. Journal of Aesthetic Education* 34, no. 3/4 (2000): 154.

patterns.[14] The fact that music increases mathematical skills makes it clear that there is a deep rooted connection between them. These are two subjects that should be explored together, not separately. One of the greatest minds of the 20th century, Albert Einstein is a basis for exploring both subjects. Einstein is famous for his theories of relativity, but a lesser known fact about him is that he was a violin player. He took lessons from an early age and even played with other physicists such as Max Planck. Einstein enjoyed playing Mozart and believed that Mozart did not create his music but instead simply discovered it already made.[15] Einstein's belief that Mozart discovered his music creates an idea that both music and math are products of nature, which would explain the multitude of connections between the two.

Conclusion

Is music just a bunch of numbers? Next time you're listening to your favorite song, take a step back and think about all the sound waves, pitches, intervals, harmonics, and overtones that are simultaneously working together to create the musical experience in your ears. Both music and math are extraordinarily complex subjects, but they are also beautiful products of nature and should not be viewed in isolation. The fact that there are such prominent connections between them makes it clear that the intersection between them is vast. While the concepts in this paper are simple mathematically, there are so many other ways music can be explored through math such as a branch of mathematics known as group theory, where there is a set of something and an operation associated with that set. A direction for further research is to treat each pitch as an object in a set and addition as the operation (where adding notes is playing multiple at the same time) and seeing what musical insight that provides. By taking musical phenomena and trying to

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[14] Holmes phd (page 5) Sylwia Holmes, *The Impact of Participation in Music on Learning Mathematics*. PhD diss., University College London, 5.

[15] Peregrine White, *Albert Einstein: Violinist*. *American Music Teacher* 31, no. 4 (1982): 24.

describe them mathematically the same way Pythagoras did, we may be able to pioneer the next direction in music and music theory.

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