Logarithmic Differentiation

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1 Motivation

The reason for this paper is to present an intuitive approach on how logarithmic differentiation works and some general cases of it. Often in textbooks and classes, this topic is overlooked despite its power.

2 Introduction

Throughout the study of calculus, several types of function arise. Knowing how to take the derivative of said functions is imperative and allows applications and insight into those functions. When presented with a function that involves a function raised to the power of other functions. These functions are in the form:

$$h(x) = f(x)^{g(x)}$$

This is not to be confused with composite functions such as e raised to the power of $sin(x)$ where chain rule can be applied. An example of a function where logarithmic differentiation is needed is:

$$f(x) = x^x$$

This is where the power of logs comes into play and allows us to find the derivative of this otherwise counterintuitive function.

3 Detailed Walkthrough

To find the derivative of $f(x) = x^x$, some log properties that become useful are:

$$log(x^a) = alog(x) \tag{1}$$
$$log(ab) = log(a) * log(b) \tag{2}$$

Writing $f(x)$ as y creates more intuition into this method.

$$y = x^x$$

Step 1: Take the log of both sides of the "equation".

$$ln(y) = ln(x^x)$$

Step 2: Use the property in equation 1 to bring the power of the argument down to a coefficient.

$$ln(y) = xln(x)$$

Step 3: Use implicit differentiation to solve for dy/dx.

$$\frac{1}{y}\frac{dy}{dx} = (1)(ln(x)) + (x)(\frac{1}{x})$$
$$\frac{1}{y}\frac{dy}{dx} = ln(x) + 1$$
$$\frac{dy}{dx} = \frac{1}{y}(ln(x) + 1)$$

Step 4: Substitute y into dy/dx
Plugging in $y = x^x$:

$$\boxed{\frac{dy}{dx} = x^x(ln(x) + 1)}$$

4 General Cases

The process in the previous section can be repeated for general cases.

1. One of the cases is: $f(x) = (x + c)^x$ where $c \in \mathbb{R}$.

   The general solution for this is:

$$\boxed{\frac{dy}{dx} = (x + c)^x (ln(x + c) + \frac{x}{x + c})}$$

2. Another case is when $f(x) = x^{cx}$ where $c \in \mathbb{R}$.

   The general solution is:

$$\boxed{\frac{dy}{dx} = x^{cx}(cln(x) + c)}$$

3. Another one of the most important cases is: $f(x) = x^{g(x)}$

   The general solution is:

$$\boxed{\frac{dy}{dx} = x^{g(x)}(g'(x)ln(x) + \frac{g(x)}{x})}$$

These are just a few general cases but there are many more. Logarithmic differentiation is one of the most powerful tools that allow us to find the derivative of confusing functions.