Lemma: Lower Bound by Domain Inclusion

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1 Proof of the Lemma

Theorem 1 *Let*

$$\textit{(IP)} \quad z^*_{IP} = \max\{c^T x : Ax \leq b,\ x \geq 0,\ x \in Z^n\}$$

and let its LP relaxation be

$$\textit{(LP)} \quad z_{LP} = \max\{c^T x : Ax \leq b,\ x \geq 0,\ x \in R^n\}.$$

Then $z_{LP} \geq z^*_{IP}$.

Define the two feasible regions:

$$F_{IP} = \{x \in Z^n : Ax \leq b,\ x \geq 0\}, \qquad F_{LP} = \{x \in R^n : Ax \leq b,\ x \geq 0\}.$$

Step 1: $F_{IP} \subseteq F_{LP}$**.**

Let $x' \in F_{IP}$ be arbitrary. Then x' satisfies $Ax' \leq b$, $x' \geq 0$, and $x' \in Z^n$. In particular, x' satisfies $Ax' \leq b$ and $x' \geq 0$. Since $Z^n \subseteq R^n$, we also have $x' \in R^n$. Hence x' satisfies every constraint defining F_{LP}, so $x' \in F_{LP}$.

Since x' was arbitrary, $F_{IP} \subseteq F_{LP}$.

Step 2: Optimizing over a superset cannot decrease the optimal value.

Let $x^* \in F_{IP}$ be an optimal solution to (IP), so that $c^T x^* = z^*_{IP}$. Such x^* exists by assumption that (IP) is feasible and bounded.

By Step 1, $x^* \in F_{IP} \implies x^* \in F_{LP}$.

Since z_{LP} is defined as the maximum of $c^T x$ over all $x \in F_{LP}$, and x^* is one particular feasible point of F_{LP}, it follows that

$$z_{LP} = \max_{x \in F_{LP}} c^T x \ \geq\ c^T x^*.$$

Step 3: Conclude.

Combining $c^T x^* = z^*_{IP}$ with Step 2:

$$z_{LP} \geq c^T x^* = z^*_{IP}.$$

Remark 1 (Minimization case) *If instead* $z^*_{IP} = \min\{c^T x : x \in F_{IP}\}$ *and* $z_{LP} = \min\{c^T x : x \in F_{LP}\}$*, the identical argument (with* min *replacing* max *and the inequality in Step 2 reversed) gives*

$$z_{LP} \leq z^*_{IP},$$

i.e. the relaxation value is a lower *bound rather than an upper bound. The direction of the inequality is entirely determined by whether the optimizer is a* max *or a* min *over the (weakly) larger set* $F_{LP} \supseteq F_{IP}$ *— the containment argument in Steps 1–2 is identical in both cases.*

Key facts: integer programs versus linear programs

Proposition 1 (Feasible-region containment) *Let* $F_{IP} = \{x \in Z^n : Ax \leq b,\ x \geq 0\}$ *and* $F_{LP} = \{x \in R^n : Ax \leq b,\ x \geq 0\}$*. Then*

$$F_{IP} \subseteq F_{LP}.$$

Every $x \in F_{IP}$ satisfies $Ax \leq b$, $x \geq 0$, and (trivially) $x \in R^n$ since $Z^n \subset R^n$. Hence $x \in F_{LP}$.

Proposition 2 (The relaxation bound) *For a maximization problem,*

$$z_{LP} \geq z^*_{IP}.$$

For a minimization problem, the inequality reverses: $z_{LP} \leq z^*_{IP}$*.*

Immediate from feasible-region containment: optimizing over the larger set $F_{LP} \supseteq F_{IP}$ can only weakly improve the optimal value in the direction of optimization.

Remark 2 (Rounding is neither valid nor safe) *There is no general relationship between the LP-optimal solution* x_{LP} *and the IP-optimal solution* x^*_{IP} *beyond the bound above. In particular:*

- *Rounding* x_{LP} *to the nearest integer point may be* infeasible *(violate a constraint entirely).*
- *Even when a rounded point is feasible, its objective value can differ from* z^*_{IP} *by an amount that grows without bound as the problem is scaled — i.e., no fixed accuracy guarantee exists for naive rounding.*

Definition 1 (Integrality gap) *The* integrality gap *of an integer program is*

$$gap = \left| \frac{z_{LP} - z^*_{IP}}{z^*_{IP}} \right|.$$

A gap of 0 *means the LP relaxation already solves the IP exactly.*

Theorem 2 (Total unimodularity closes the gap) *Let A be the constraint matrix of the LP relaxation. If A is* totally unimodular *(every square submatrix of A has determinant 0, 1, or -1) and b is integer-valued, then every vertex (basic feasible solution) of F_{LP} is integer-valued. Consequently*

$$z_{LP} = z^*_{IP},$$

and the integrality restriction may be dropped entirely — the LP relaxation solves the IP.

Remark 3 *Classic examples with totally unimodular constraint matrices include network-flow problems, the transportation problem, and the bipartite assignment problem. This is why these particular "IP-flavored" problems are, in practice, solved as ordinary LPs.*

Proposition 3 (Branch-and-bound convergence) *Let $\underline{z}$ denote the incumbent (best integer-feasible objective value found so far) and let $\overline{z}$ denote the best (least restrictive) bound among all unexplored subdivisions in the enumeration tree, for a maximization problem. Then at every stage of branch and bound,*

$$\underline{z} \le z^*_{IP} \le \overline{z},$$

and the algorithm terminates with $\underline{z} = \overline{z} = z^*_{IP}$ after finitely many subdivisions (since F_{IP} is finite).

Proposition 4 (Branching preserves the integer feasible set) *Let L_j be a subdivision with LP-optimal solution containing a fractional coordinate $x_k = v \notin Z$. Define the two child subdivisions*

$$L'_j = L_j \cap \{x_k \le \lfloor v \rfloor\}, \qquad L''_j = L_j \cap \{x_k \ge \lceil v \rceil\}.$$

Then

$$F_{IP} \cap L_j = \left(F_{IP} \cap L'_j\right) \cup \left(F_{IP} \cap L''_j\right),$$

i.e., no integer-feasible point of L_j is lost by branching, while the fractional point $x_k = v$ is excluded from both children.

Since $v \notin Z$, no integer lies in the open interval $(\lfloor v \rfloor, \lceil v \rceil)$. Hence every integer point in L_j satisfies $x_k \le \lfloor v \rfloor$ or $x_k \ge \lceil v \rceil$, so it lies in L'_j or L''_j. Conversely $L'_j \cup L''_j \subseteq L_j$. The point $x_k = v$ itself satisfies neither $x_k \le \lfloor v \rfloor$ nor $x_k \ge \lceil v \rceil$, so it is excluded from both.

Remark 4 (Geometric structure) *An LP optimum, when it exists and is finite, always occurs at a* vertex *(extreme point) of the polytope F_{LP} — this is what makes simplex, which only ever examines vertices, complete. An IP optimum, by contrast, is a vertex of the* integer hull $\mathrm{conv}(F_{IP})$ *(the convex hull of the integer feasible points), which is generally a strictly smaller polytope nested inside F_{LP}. Cutting-plane methods work by adding valid inequalities that shrink F_{LP} toward* $\mathrm{conv}(F_{IP})$ *without ever cutting away a point of F_{IP} itself.*

Remark 5 (Complexity contrast) *Linear programming is solvable in polynomial time (e.g. via interior-point methods; simplex is efficient in practice though not worst-case polynomial). Integer programming is* NP*-hard in general — no algorithm is known (or, under* $\mathsf{P} \neq \mathsf{NP}$*, exists) that solves all instances in time polynomial in* n*. This is the fundamental reason IP relies on LP relaxation plus search (branch and bound, cutting planes) rather than a direct polynomial algorithm: the relaxation is the tractable core around which the intractable search is organized.*