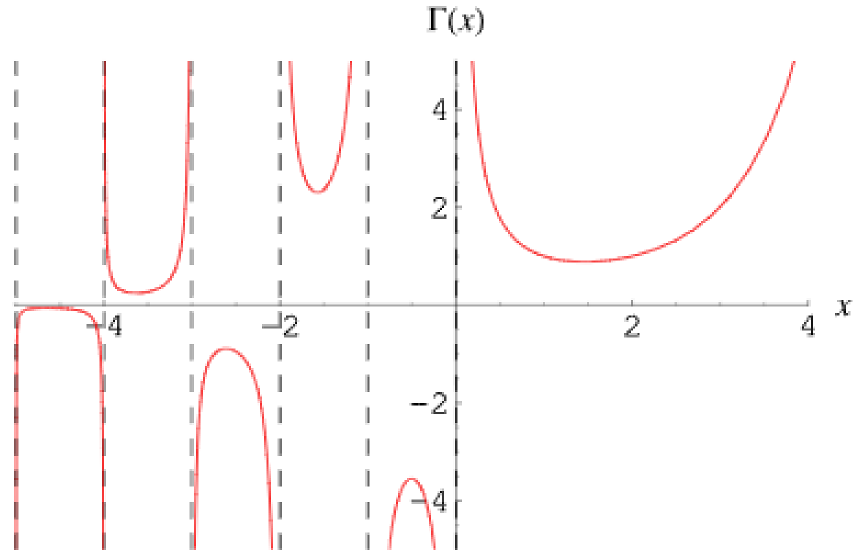
Derivative of a Factorial



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1.1 Definition of a Factorial

In math, a factorial, written as n!, is the multiplication of all numbers between n and 1. In this case, n must be a positive integer.

Example 1. 4! $= 4 \times 3 \times 2 \times 1$

The result of this operation is simply 24.

1.2 Definition of a Function for Non-Real Parameters

While factorials are only able to be applied to positive integers, there is a function that defines the factorial for non-real or imaginary numbers. This is known as the Gamma Function.

$$\Gamma(z-1) = \int_0^\infty t^{z-1} e^{-t} dt = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-1} e^{z/k}, \qquad \gamma \approx 0.577216$$

For this paper, we only need the first part of this function.

$$\Gamma(z-1) = \int_0^\infty t^{z-1} e^{-t} dt$$

This function allows us to take imaginary parameters and has recursive properties. This function is simple $(z-1)$!

1.3 Differentiating an Integral

Before we can differentiate the Gamma Function, we need to understand how to differentiate an integral.

Example 2.

$$\int_a^b f(x,t)\, dt$$

Now if we were to differentiate with respect to x:

$$\frac{d}{dx} \int_a^b f(x,t)\, dt$$

The derivative will now become part of the integral as a partial derivative:

$$\int_a^b \frac{\partial}{\partial x} f(x,t)\, dt$$

This allows any “t” variable to be moved outside of the integral.

1.4 Differentiation of the Gamma Function

With the knowledge from 1.3, we can apply that to the Gamma Function:

$$\Gamma(z-1) = \int_0^\infty t^{z-1} e^{-t} dt$$

Taking the non-recursive version of this function:

$$\Gamma(z) = \int_0^\infty t^z e^{-t} dt$$

*Note, this is an improper integral. The proper notation of this integral is:

$$\lim_{\alpha \to \infty} \int_0^\alpha t^z e^{-t} dt$$

Now, we can differentiate:

$$\Gamma'(\mathrm{z}) = \frac{d}{dz} \int_0^\infty t^z e^{-t} dt$$

$$\Gamma'(z) = \int_0^\infty \frac{\partial}{\partial z} t^z e^{-t} dt$$

Since e^{-t} is now considered a constant, the expression now becomes:

$$\Gamma'(z) = e^{-t} \int_0^\infty \frac{\partial}{\partial z} t^z dt$$

Now we simply take the partial derivative of t^z with respect to z:

$$\Gamma'(z) = e^{-t} \int_0^\infty \frac{\partial}{\partial z} t^z \ln(t) dt$$

Resulting in our final answer:

$$\Gamma'(z) = e^{-t} \lim_{\alpha \to \infty} \int_0^\alpha \frac{\partial}{\partial z} t^z \ln(t) dt$$