PROOF OF EULER'S IDENTITY AND FORMULA

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1. Introduction

One of the most famous equations in all of mathematics is Euler's Identity, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. This has applications in all fields of math as well as physics and engineering and allows to represent complicated functions with both real and complex parts as a sum of the real parts and imaginary parts.

The focus of this paper is to prove Euler's formula and identity in order to provide intuition into where it came from and how it can be used. There are many applications including systems of differential equations, which will be demonstrated later in this paper.

2. Proof

Recall that a function can be replicated at an x coordinate a (such that $a \in \mathbb{R}$) using a function and its derivatives, as long as a function is continuous and has n derivatives on an interval. Functions that have Taylor Series are contained in the set:

$$S = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is infinitely differentiable at } a,\, a \in \mathbb{R}, f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n\}$$

Since e^x, $\sin x$, and $\cos x \in S$, Taylor Series of these functions can be made, using polynomials to approximate them. While not all functions have a Taylor Series that converges at all values, e^x, $\sin x$, and $\cos x$ all have a radius of convergence that equals infinity.[1] The Taylor series for e^x, $\sin x$, and $\cos x$ are as follows.

(1) e^x centered at $x = 0$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!}$$

(2) $\sin(x)$ centered at $x = 0$

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(3) $\cos(x)$ centered at $x = 0$

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!}$$

Proof of Euler's Formula. Substituting x with ix, we obtain a Taylor Series for e^ix. $e^ix = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \ldots \frac{(ix)^n}{n!}$. Simplifying the powers of i using $i^2 = -1$ allows us to rewrite the identity.

Simplifying gives us $e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} + \frac{x^6}{6!}$. From this, one can see that the terms with an even power of x do not have an imaginary part and the odd powered terms contain a factor of i. Separating this equation by the real and imaginary parts gives us: $e^{ix} = [1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!}] + i[x - \frac{x^3}{3!} + \frac{x^5}{5!} = \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}]$.

The Taylor expansion of e^{ix} results in the expansions for $\sin x$ and $\cos x$ appearing, with cos occupying the real part and sin having an imaginary factor i. The final

[1] Appendix A

identity can now be written: $\boxed{e^{ix} = \cos x + i \sin x}$. It is also commonly written in terms of θ instead of x due to its use in complex analysis: $e^{i\theta} = i\sin\theta + \cos\theta$.

Using Euler's Identity, Euler's Formula ($e^{i\pi} + 1 = 0$) can be derived. Plugging in π to $e^{i\theta} = \cos\theta + i\sin\theta$: $e^{i\pi} = \cos\pi + i\sin\pi$. Therefore, $e^{i\pi} = -1$, or more commonly written as $\boxed{e^{i\pi} + 1 = 0}$. □

3. Application in Systems of Differential Equations

Consider the following system of differential equations:

$$\frac{dx_1}{dt} = -x_2,$$
$$\frac{dx_2}{dt} = x_1.$$

Representing the system with a matrix A:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We can solve this system by finding the eigenvalues and eigenvectors. The characteristic polynomial for this matrix is:

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}.$$

Expanding the determinant gives:

$$p(\lambda) = \lambda^2 + 1.$$

Setting $p(\lambda) = 0$ gives the eigenvalues:

$$\lambda = \pm i.$$

The eigenvector for $\lambda_1 = i$ is:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

If $a + bi$ is an eigenvalue with eigenvector $\vec{v}_1 = \vec{r} + i\vec{s}$, then $a - bi$ is also an eigenvalue with eigenvector $\vec{v}_2 = \vec{r} - i\vec{s}$. Using this, we find the eigenvector for $\lambda_2 = -i$:

$$\vec{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Note that the matrix representing the coefficients of the system of differential equations is non-defective because the algebraic multiplicity equals the geometric multiplicity for all λ_i. In other words, each eigenspace is spanned by its eigenvector, and the eigenvectors from each eigenvalue form a basis for the eigenspace.

Now that we have the eigenvalues and eigenvectors for this system, we can obtain both solutions using just one of the eigenvalues and its corresponding eigenvector, thanks to Euler's Identity.

The general solution to this system is:

$$\mathbf{y}(t) = c_1 e^{it} \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

However, due to Euler's Identity, we can use only one of the eigenvalues and eigenvectors to derive two solutions. Using $\lambda_1 = i$ and $\vec{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$, we separate the eigenvector into its real and imaginary parts:

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Using Euler's Identity, we can rewrite e^{it} as $\cos(t) + i\sin(t)$. Grouping the real part of the eigenvector with the cos term and the imaginary part with the sin term we get:

$$y(t) = \cos(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The solution matrix to this system of differential equations is:

$$y(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

4. Final Thoughts

This has many extensions and allows us to view many concepts that weren't otherwise tangible, such as the log of a negative number. Consider $\ln(-1)$. This is something that is undefined, but using Euler's Formula, we can define it. Using $e^{i\pi} = -1$, we can take the natural log of both sides resulting in: $i\pi = \ln(-1)$. Euler's Identity and Formula are very powerful and have applications in many fields.

Additionally, Euler gave us a way of reimagining the complex plane. Consider the real axis as the x-axis and the complex axis as the y-axis. Since we know from the unit circle (circle with $r = 1$) that the Cartesian plane can be thought of in terms of trigonometric function: sin and cos, we can extend this to a complex case. Even though Euler's Identity $e^{i\theta} = \cos\theta + i\sin\theta$, is not intuitively visual, representing it on a complex plane using a circle allows for even more intuition into where it came from and how it can be used.

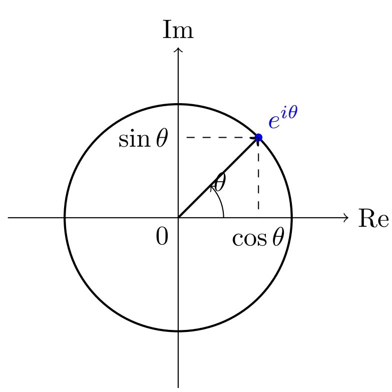

Figure 1. Illustration of $e^{i\theta}$ in the complex plane.

As seen in Fig. 1, Euler's Identity is very powerful and allows us to describe complex functions. By being able to represent complex functions visually using functions we are comfortable with, Euler allowed us to further study complex functions, which have many applications in the real world.

Appendix A. Radius of Convergence of Taylor Series

In this appendix, we provide the proofs of the radius of convergence for the Taylor series expansions of e^x, $\sin x$, and $\cos x$.

A.1. **Taylor Series for e^x.** The Taylor series for e^x is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To find the radius of convergence, we apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since the limit is 0, the series converges for all x, implying the radius of convergence is $R = \infty$.

A.2. **Taylor Series for $\sin x$.** The Taylor series for $\sin x$ is:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Using the ratio test, we consider the general term:

$$a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

The ratio is:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)} = 0.$$

Thus, the radius of convergence is $R = \infty$.

A.3. **Taylor Series for $\cos x$.** The Taylor series for $\cos x$ is:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Similarly, using the ratio test:

$$a_n = \frac{(-1)^n x^{2n}}{(2n)!}.$$

The ratio is:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{2(n+1)}}{(2(n+1))!}}{\frac{x^{2n}}{(2n)!}} \right| = \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+1)} = 0.$$

Again, the radius of convergence is $R = \infty$.