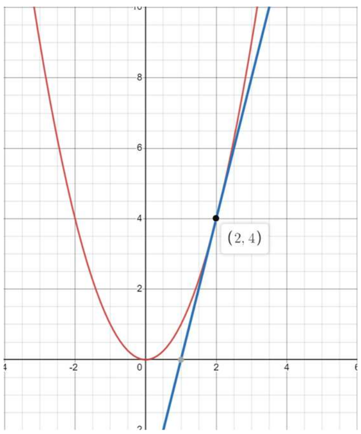
Chapter 2 – The Derivate and Rules of Differentiation

Now that we understand limits, we can move on to one of the main ideas of calculus: the derivative.

In very simple terms, the derivative is the slope of a function. There are different interpretations of what a derivative is, but this text uses a geometric interpretation.



The derivative is the slope of the tangent line. In figure 11, the blue line is the tangent line to x^2 at the point (2,4). The derivative is simply the slope of the blue line.

Fig 11. Graphical meaning of a derivative

The derivative can be formally defined using limits:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This definition is taking a tiny change between f(x) and f(x+h) and taking the limit as that change goes to 0. This is finding the slope at an infinitesimally small point.

Suppose we have a function $f(x) = x^2$ and we wanted to find the derivative of that function, we would use the definition of the derivative.

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

Expanding the binomial gives us:

$$\lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

Simplifying:

$$\lim_{h \to 0} (2x + h)$$

Now, we can evaluate the limit giving us:

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = 2x$$

Therefore, the derivative of x^2 is $2x$.

This is the generic derivative for the function $f(x) = x^2$, if we wanted the actual slope at a point, we simply plug in the x value of that point into the derivative. For example, the slope at x=2 is 4 because the derivative of x^2 is 2x and 2(2)=4.

Luckily, there is a much easier to evaluate derivatives of polynomials. This is done through the **power rule**. The power rule is a only used to find the derivative of a polynomial.

The power rule states:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

*Note n is any real number.

The operator, $\frac{d}{dx}$ is the derivative operator, which is read as "the derivative with respect to x". This just means that we are taking the derivative of the x variable and no others. If there were other variables, they would have to be explicitly stated but we'll get to this later on in the text.

Now if we were to take the same derivative as before but use the power rule we get:

$$\frac{d}{dx}(x^2) = 2x^1 = 2x$$

Just like limits, derivatives have helpful properties that we use when evaluating.

Property Name	Definition
Derivative of a constant	$\frac{d}{dx}(c) = 0$
The Sum Rule	$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
The Difference Rule	$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$
The Constant Multiple Rule	$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}(f(x)$

Table 5. Derivative Properties

Property Name	Example
Derivative of a constant	$\frac{d}{dx}(5) = 0$
The Sum Rule	$\frac{d}{dx}[x^2 + 3x] = 2x + 3$
The Difference Rule	$\frac{d}{dx}[2x^2 - 4x] = 4x - 4$

The Constant Multiple Rule	$\frac{d}{dx}[3x^2] = 3\frac{d}{dx}x^2$

Table 6. Examples of Derivative Properties

At this point, we can now evaluate derivatives of polynomials and use the properties. The power rule also works for fractional powers, such as square roots and cubed roots.

For example, if we have the function $f(x) = \sqrt{x}$, we can rewrite that as $f(x) = x^{1/2}$ and use the power rule: $f'(x) = \frac{1}{2}x^{-1/2}$.

*Note that $f'(x)$ and $\frac{d}{dx}$ have the same function. They are different notations for the derivative, but both represent a function being differentiated.

The Product Rule and Quotient Rule

If we wanted to take the derivative of a function that was the product of two functions, we would use the product rule. The product rule says the derivative of $f(x)g(x) = f'(x)g(x) + f(x)g'(x)$.

Example of the product rule:

$$\frac{d}{dx}xsin(x) = (1)\sin(x) + (x)\cos(x)$$

The quotient rule is used when taking the derivative of a function that is the quotient of two other functions. The derivative of a quotient of functions is $\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g\prime(x)}{g(x)^2}$.

Example of the quotient rule:

$$\frac{d}{dx}\left(\frac{x}{\sin(x)}\right) = \frac{(1)((\sin(x)) - (x)(\cos(x))}{(\sin(x))^2}$$

The Chain Rule

Next up, we will examine how to take the derivative of a composite function.

Suppose we have some composite function: $f(g(x))$, the derivative of this function is: $f'(g(x)g'(x)$. When taking the derivative of a composite function, we must take the derivative of the inside (g(x)) and multiply it by the derivative of the outside (f'(g(x)).

Example of the chain rule:

$$\text{f(x)} = \sin(2x)$$

For this function, f(x), there are two functions present: sin(u) and 2x. The derivative of sin(u) is cos(u) and the derivative of 2x is 2. In this case, u is 2x, giving us:

$$f'(x) = \cos(2x) \cdot 2$$

Derivative of e^x and $ln(x)$

Now, we will look at the derivative of these two functions.

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}$$

This means that when we take the derivative of e^u, it is equal to e^u times the derivative of 'u'.

$$e^{\sin(x)} = e^{\sin(x)}\cos(x)$$

$$\frac{d}{dx}(\ln(u)) = \frac{1}{u}\frac{du}{dx}$$

Again, we take the reciprocal of the log argument and multiply by the derivative of the argument.

$$\frac{d}{dx}(\ln(\sin(x)) = \frac{1}{\sin(x)}\cos(x) = \cot(x)$$

Slope at a Point

One application of the derivative is it tells us the slope at any point of a function. For example, if we have the function $f(x) = x^3$, to find the slope at any point of this function, we take the derivative and plug the point into the derivative.

$$f'(x) = 3x^2$$

$$f'(2) = 3(2)^2 = 12$$

This tells us the slope at x=2 is 12.

Finding the Equation of a Tangent Line

Using the knowledge of the previous section, we can find the equation of a tangent line at a point on a function. A tangent line is a line that touches a function at only one point. To find these equations, we will use point-slope form.

Point-Slope Form:

$$y - y_1 = m(x - x_1)$$

*m is the slope, (x_1, y_1) is the point

Let's say we want to find the equation of the tangent line of sin (x) at $x = \pi/2$

First, we have to find the slope, by taking the derivative and plugging in the x coordinate.

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\cos\left(\frac{\pi}{2}\right) = 0$$

Therefore, the slope at this point is 0. The sin of $\frac{\pi}{2}$ is 1. We can now write our equation for the tangent line.

$$y - 1 = 0(x - \frac{\pi}{2})$$

$$y = 1$$

This makes sense graphically:

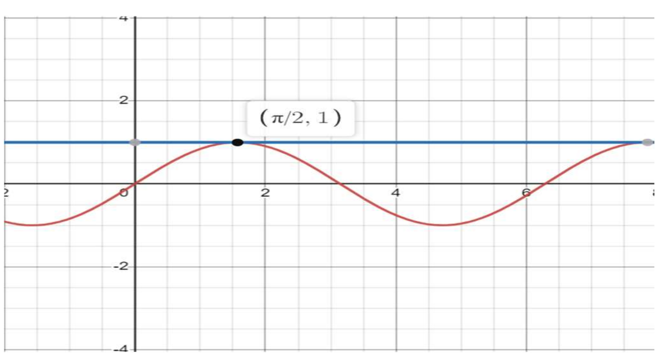


Fig 12. Tangent Line of sin(x)

Important Derivatives to Know

$$\frac{d}{dx}x^n = \mathrm{n}x^{n-1}$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\tan(\mathrm{x}) = sec^2(x)$$

$$\frac{d}{dx}\sec(x) = \sec(\mathrm{x})\tan(\mathrm{x})$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\cot(x) = -csc^2(x)$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

*For all of these don't forget to use the chain rule

Implicit Differentiation

When we have a function of several variables, we need to **imply** what we are taking the derivative with respect to.

For example, if we want to take the derivative of the equation of a circle: $x^2 + y^2 = 1$, we need to imply what we want to take the derivative with respect to.

If we wanted to find $\frac{dy}{dx}$:

$$\frac{dy}{dx}(x^2 + y^2 = 1)$$

For the terms with x, we can take the normal derivative. However, for the terms with y, we need to apply the chain rule and multiply by $\frac{dy}{dx}$, which is what we are trying to find.

$$2x + 2y\frac{dy}{dx} = 0$$

Now, to solve, we isolate $\frac{dy}{dx}$.

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Steps for Implicit Differentiation

1. Take the derivative with respect to the variable in the denominator of the differential.
2. Multiply any variable that isn't implicit by the derivative.
3. Move all terms with the derivative to one side.
4. GCF out the derivative and isolate.