Chapter 1 – Limits and Continuity

To understand calculus, we have to know what a limit is and how to evaluate them. Limits are written as: $\lim_{x\to a} f(x)$. This is read as "the limit as x approaches a of f(x). In this case a is an arbitrary number. Limits help us find the actual value of a function at a point.

First, we need to know whether a function is continuous or discontinuous. A function is continuous if you can sketch it without having to pick up your pen. If you must pick up your pen, the function is discontinuous.

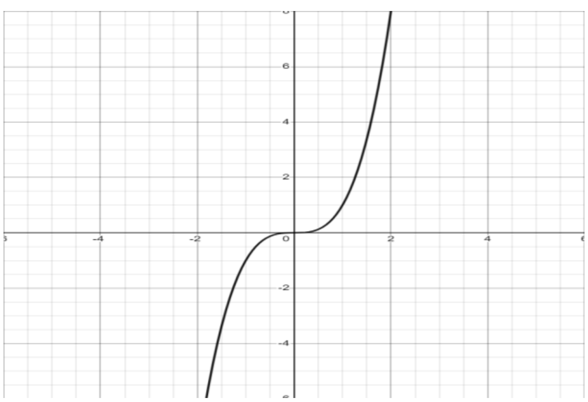


Fig 1. Graph of a continuous function (x^3)

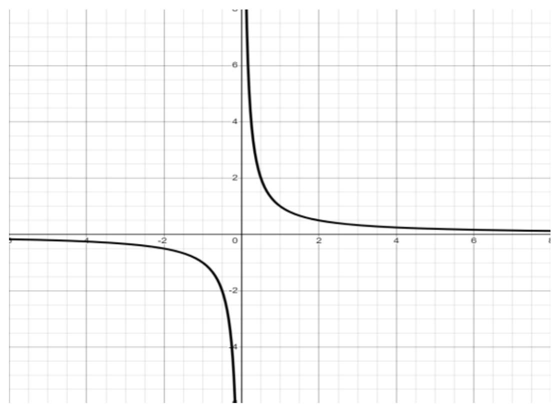


Fig 2. Graph of a discontinuous function

As seen in Figure 1, this function can be graphed without picking up your pen, whereas Figure 2 cannot. Figure 1 shows a function that would be continuous, and Figure 2 shows a function that is not continuous. There are a handful of types of discontinuities. The first is known as a **vertical asymptote**. A vertical asymptote is a vertical line where the function will have no values. Figure 2 shows a vertical asymptote at x=0. The way we find vertical asymptotes is to see where a function would be equal to 0 in the denominator.

For example, we have the function $f(x) = \frac{1}{(x-3)}$. The way we find the vertical asymptote is by setting the denominator equal to zero.

$$x - 3 = 0$$

$$x = 3$$

The vertical asymptote for this function is x=3. Vertical asymptotes are always found this way. The next time of discontinuity is a **hole**. This is considered a removable discontinuity. This means that there can't be a value at this point. This is found by being able to cancel something out. For example:

$$f(x) = \frac{(x-3)}{(x+3)(x-3)}$$

In this case, we can cancel out (x-3). This means there is a hole at x=3. We can find the f(x) value of this hole by removing the (x-3) term and plugging 3 into the new function that we'll call g(x).

$$g(x) = \frac{1}{(x+3)}$$

Plugging x=3 into our new function we get $g(3) = 1/6$. This means we have a hole at (3, 1/6).

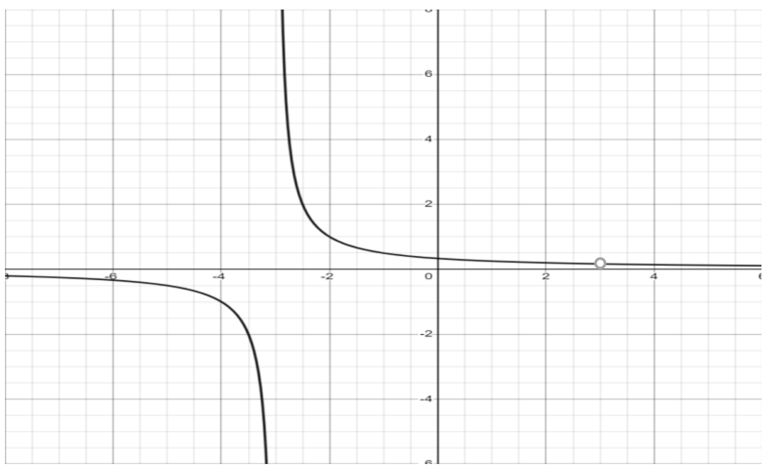


Fig 3. The graph of f(x) with a hole at x=3.

As we can see from this graph, there is a hole at x=3. We use the new function g(x) to find the f(x) value of the hole since we cannot directly substitute the x value of the hole into the original function, f(x).

Now that we understand the two types of discontinuities (vertical asymptotes and holes), we can move onto finding limits. For a limit to exist, the left-handed and right-handed limit have to be equal to each other.

$$If\colon \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

$$Then\colon \lim_{x \to a} f(x) = c$$

$* \, Note\colon c \text{ is an integer or } \pm\infty.$

If you're confused about this, don't worry. Limits have a "left-hand" and a "right-hand". All this means is the value of the limit from the left of the point and from the right of the point. For a limit to exist, the left-hand and the right-hand must have the same value.

Let's look at this through a simple example. Let's look at f(x)=x.

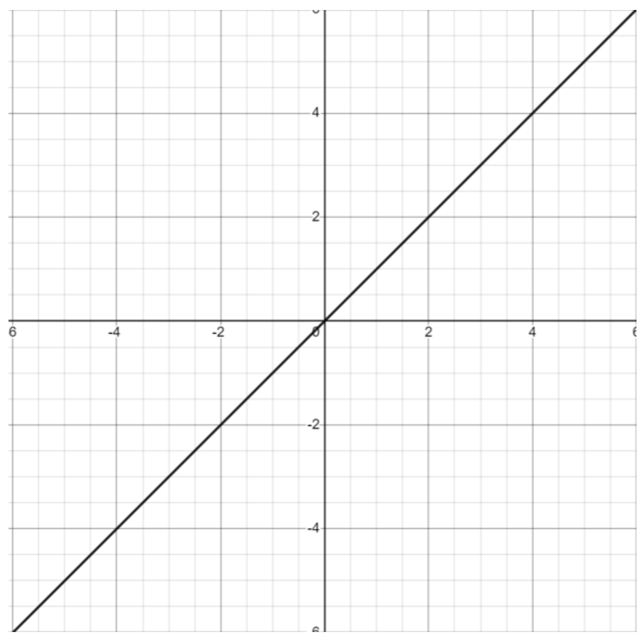


Fig 4. Graph of f(x)=x.

Let's find the limit as x approaches 3 or: $\lim_{x \to 3} x$. We can look at the graph and know that this limit is equal to 3. However, for this to be true, the limits from the left and right have to both be 3. If we look from the left, we can take a point very close to three like 2.99999. This limit is extraordinarily close to 3 and since this will keep on going, it will get to 3. The same goes for the right side. If we take the limit as x approaches 3.000001, the limit will be equal to 3. Since both limits equal 3, $\lim_{x \to 3} x = 3$. For limits like this, there is a much simpler way of evaluating them. The simplest way of evaluating limits is by **direct substitution**. Direct substitution simply means plug in the number the limit is approaching into the function.

$$\lim_{x \to 4} x^2$$

$$f(4) = (4)^2$$

$$\lim_{x \to 4} x^2 = 16$$

The next method for evaluating limits is by rationalizing. Rationalizing is getting rid of a radical by multiplying the numerator and denominator by the radical. The way you know you need to rationalize is if you try direct substitution and it fails. It fails if you get an **indeterminate form**. Examples of indeterminate forms are $\frac{0}{0}, \frac{\infty}{\infty}, and\ \frac{-\infty}{-\infty}$. If you get an indeterminate form, it means the method you're using to solve can't solve said problem.

What happens when we have:

$$\lim_{x \to 25} \frac{5 - \sqrt{x}}{25 - x}$$

If we try to directly substitute 25 in for x, we get 0/0 which is an indeterminate form. This tells us that we can't solve this limit this way, but it can be solved another way. We can solve this limit by rationalizing.

$$\lim_{x\to 25}\frac{5-\sqrt{x}}{25-x}\cdot\frac{(5+\sqrt{x})}{(5+\sqrt{x})}$$

Multiplying by the conjugate pair of $5-\sqrt{x}$ will get rid of the square root.

$$\lim_{x\to 25}\frac{(25-x)}{(25-x)(5+\sqrt{x})}$$

This then gives us:

$$\lim_{x\to 25}\frac{1}{5+\sqrt{x}}$$

Now, we can directly plug-in 25 to the limit.

$$\frac{1}{5+\sqrt{25}}=\frac{1}{10}$$

Therefore,

$$\lim_{x\to 25}\frac{5-\sqrt{x}}{25-x}=\frac{1}{10}$$

Another way of solving limits that don't work with direct substitution is by factoring.

$$\lim_{x\to 1}\frac{x^3-1}{x-1}$$

If we plug in 1, we get 0/0. However, we can factor the numerator by using the difference of two perfect cubes. Giving us:

$$\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}$$

We can now reduce and directly substitute x=1.

$$\lim_{x\to 1}(x^2+x+1)$$

$$(1)^2+1+1=3$$

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=3$$

(These properties are very notation heavy. They are very simple when you use them so try not to get overwhelmed by the notation of them.)

Next up are some important properties of limits we need to know.

1. $\lim_{x\to a} c = c \textit{ where c is a constant}$
2. $\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
3. $\lim_{x\to a} f(x) - g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
5. $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x) \textit{ where c is a constant}$

Property number five is known as the constant multiple rule and will be seen again as we continue our study of calculus.

Example of property 1.

$$\lim_{x\to 3} 6 = 6$$

Since f(x)=6 is simply a straight horizontal line, the limit at any point will be 6. This is the same for the limit of any horizontal line.

Example of property 2.

$$\lim_{x\to 1} x + x^2 = \lim_{x\to 1} x + \lim_{x\to 1} x^2$$

$$= 1 + 1 = 2$$

Example of property 3.

$$\lim_{x\to 4} x - x^2 = \lim_{x\to 4} x - \lim_{x\to 4} x^2$$
$$= 4 - 16 = -12$$

Example of property 4.

$$\lim_{x\to 3} \frac{1}{x} = \frac{\lim_{x\to 3} 1}{\lim_{x\to 3} x}$$

$$= \frac{1}{3}$$

Notice how the numerator uses property number 1.

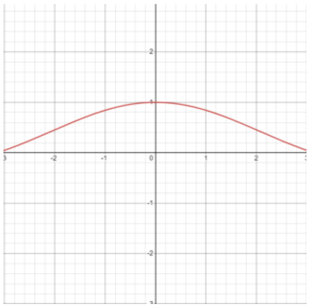
Example of property 5.

$$\lim_{x\to 2} 3x^2 = 3 \lim_{x\to 2} x^2$$

$$= 3(4) = 12$$

There are two important limit identities we need to know. The first one is:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$



As we can see from Figure 5, at x=0, f(x)=1 which verifies this limit. We can also verify this limit by taking a number very close to 0 from the left and right side. From the left, we can use -0.0001 and from the right we can use 0.0001. If we were to take values closer and closer to 0, we see from Table 1 that at x=0, the limit is equal to 1. This is another way of finding limits by using a table. We can take values that get closer and closer to the desired value to find the limit.

Fig. 5 The graph of $\frac{\sin x}{x}$.

x	***f(x)***
-0.0001	0.999
0.0001	0.999

Table 1. x and f(x) values near 0 of $\frac{\sin x}{x}$.

The other identity is: $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$

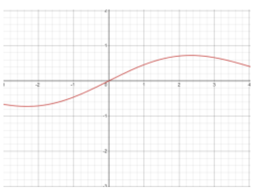


Fig 6. The graph of $\lim_{x \to 0} \frac{1-\cos x}{x}$.

x	***f(x)***
-0.0001	4.9E-05
0.0001	4.9E-05

Table 2. x and f(x) values near 0 of $f(x) \frac{1-\cos x}{x}$

As shown in the graph, the limit as x approaches zero is equal to zero. We also see in the table that when we plug in a number that is very close to zero, we get a number that is very tiny and close to zero.

Next up is evaluating limits at infinity. This is a little trickier than the previous limits we've been doing but it is not too challenging.

First off, we need to understand what a horizontal asymptote is and how we can find them. A horizontal asymptote is an imaginary horizontal line that the function will level off at as the function goes towards positive and negative infinity. Note that a horizontal asymptote can sometimes be crossed by a function, but a vertical asymptote will never be crossed. If a function crosses the horizontal asymptote, it will be near the origin of the function. As the function goes towards positive and negative infinity, it will obey the horizontal asymptote. This is important for when we take the limit as x approaches positive or negative infinity.

There are three different scenarios that will determine if a function has a horizontal asymptote.

Scenario Number	Function	Horizontal Asymptote	Example
1	$\frac{x^a}{x^b}$ *where* $a > b$	None	$f(x) = \frac{x^3}{x^2}$
2	$\frac{x^a}{x^b}$ *where* $a < b$	$y = 0$	$f(x) = \frac{x^2}{x^3}$
3	$\frac{x^a}{x^b}$ *where* $a = b$	$Ratio\ of\ a/b$	$f(x) = \frac{x^3 + x^2}{x^3}$

Table 3. Types of horizontal asymptotes

Functions that have horizontal asymptotes will always take one of those three forms. In scenario number one, this function is what we call "top-heavy", meaning that the numerator will outweigh the denominator. Since this happens, there is no horizontal asymptote. The next scenario is if the highest power of the numerator is greater than the highest power of the denominator. This is known as a "bottom-heavy" function. This has the horizontal asymptote y=0. The third scenario is if the highest power of the numerator and denominator are equal. In this case, the horizontal asymptote is the ratio of the coefficients of the highest powers.

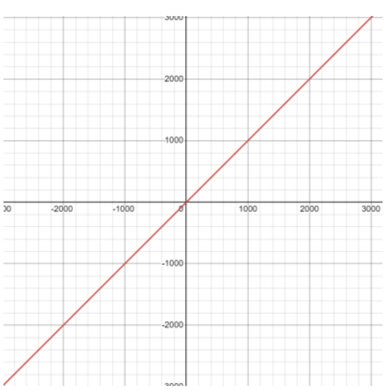


Fig 7. Example of scenario 1

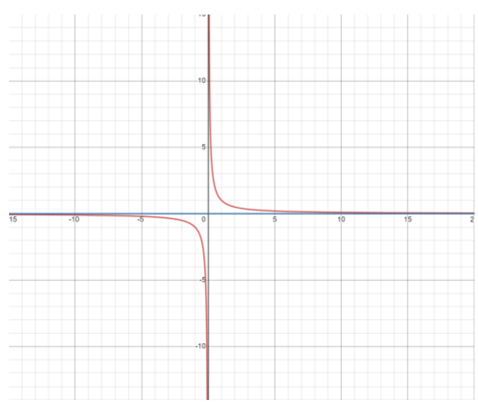


Fig 8. Example of scenario 2

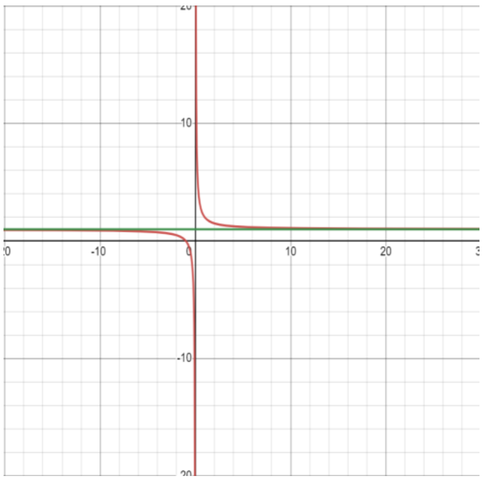


Fig 9. Example of scenario 3

As we can see from Figure 7, there is no horizontal asymptote because the function increases without bounds as it goes from negative infinity to infinity. In Figure 8, since the denominator is greater than the numerator, the horizontal asymptote is y=0. In Figure 9, the highest power of the numerator is the same as the highest power of the denominator. In this case, the asymptote is the ratio of those coefficients or y=1/1.

Now that we understand horizontal asymptotes, we can now evaluate some limits at infinity. When the limit is asking you to evaluate as x approaches positive or negative infinity for these functions, you simply need to see what the horizontal asymptote is for that function.

$$\lim_{x\to\infty}\frac{x^2}{x^3}=0$$

Since this function has a horizontal asymptote at y=0, as x approaches infinity, the limit will be equal to 0.

$$\lim_{x\to\infty}\frac{x^3}{x^2}=DNE$$

This function has no horizontal asymptote since the highest power of the numerator is greater than that of the denominator. This means that as x approaches infinity, y is increasing without bounds therefore the limit does not exist.

$$\lim_{x\to\infty}\frac{x^3}{x^3}=1$$

Similarly, to the first example, we just need to look at the horizontal asymptote to find this limit. Since the highest powers are the same, the asymptote is y=1, therefore this limit is equal to 1.

Now, what if we want to find the limit as x approaches 0 of $\frac{1}{x^2}$. In this case, none of the previous methods will work. Direct substitution will result in an indeterminate form, and we can't factor or rationalize. In this case, we will have to use a table.

x	**f(x)**

-0.0001	1E8
0.0001	1E8

As we can see from this table, when we plug in numbers really close to zero, we get large numbers. Since both sides of the limit agree, this limit is equal to infinity. However, if a limit is equal to infinity or negative infinity, the limit **does not exist (DNE)** because it increases without bounds.

Table 4. x and f(x) values near 0

The next time of limit is that a limit that approaches infinity of an exponential function. For example, if we have some exponential function a^x (where a is a constant), the limit as x approaches infinity for all positive a values, will be equal to zero if a < 1.

$$\lim_{x\to\infty} \frac{1^x}{2}$$

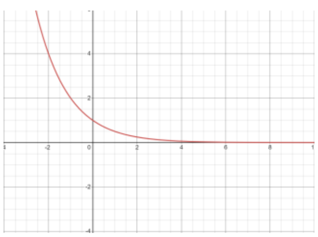


Fig 10. Graph of $(\frac{1}{2})^x$

As we can see from Figure 10, this limit quickly approaches 0. If we were to have the limit approach negative infinity, the limit would not exist.

Next up is the limit of a piecewise function. Piecewise functions are functions that are defined by several equations with bounds. Each equation has a domain restriction. Piecewise functions are written like this:

$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

This function is the same as $f(x) = |x|$. We can use piecewise functions to break up complicated functions and make them simpler. We can also take limits of piecewise functions. For example, let's say we wanted to take a limit of this piecewise function.

$$\lim_{x\to 4} f(x)$$

In this case, we need to look at the domain restriction of the piecewise to find which part of the piecewise we need to use to evaluate this limit. Since 4 is ≥ 0, we will use f(x)=x. We can simply plug in 4 directly to get:

$$\lim_{x\to 4} f(x) = 4$$

A Summary of Limits

Limits are very useful in our study of calculus and will continue to show up as go through the subject. A limit allows us to find the precise value of a function at a certain point, even if the point does not exist.

Ways to find limits:

1. Direct Substitution – Plug the number from the limit directly into the function to get what the limit is equal to.
2. Factoring – Factor the expression to try to remove a term that will make direct substitution possible.
3. Rationalize – Multiply by the conjugate pair of a square root term to try to make direct substitution possible.
4. Table of Values – if the first three ways are not possible, make a table of values where you take a number very close to the limit to see the behavior of the function at that point. If the limit is going towards ±infinity, plug in a really big positive or negative number to see the behavior of the function.