CHESS AND COMBINATORICS

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1. Introduction

A classic problem in the world of combinatorics is given an n by n chess board, try to place n non-attacking queens on the board. An interesting question arises from this: can you place $n+1$ non-attacking queens on the same n by n board? In chess, queens can move along the column, row, or diagonal, allowing them to cover a significant area. This makes it incredibly difficult to place $n+1$ queens such that they don't attack each other. Combinatorics allows us to prove mathematically why it isn't possible to place $n+1$ non-attacking queens on an n dimensional board.

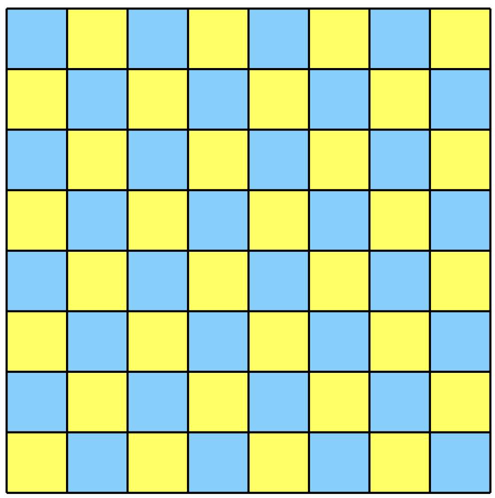

Figure 1. An 8×8 chessboard.

Consider the above 8×8 chess board. The objective is to place 9 non-attacking queens on the board above. After trying a few configurations, it becomes clear that this is not possible. However, using scenarios where something does not work is not rigorous enough to prove that it will always not work. This is where more general arbitrary allow us to prove that it will not work in every situation.

2. Pigeonhole Principle

A fundamental principle in combinatorics is the Pigeonhole Principle. This principle says that if you have two positive integers n and m, such that $n > m$, and one attempts to place n identical balls into m identical buckets, then there must be one bucket with at least two balls. This principle came from the idea that there are n pigeons (yes, like the bird) and m pigeonholes (little boxes that fit one pigeon). If there are more pigeons than pigeonholes, then there must be at least one pigeonhole with more than one pigeon. [1]

This is a very powerful principle that allows us to simplify very complex problems. Consider the following problem: There is a network of six computers. Each computer is connected to one or more other computers. Prove that there exist at least two computers connected to the same number of computers.

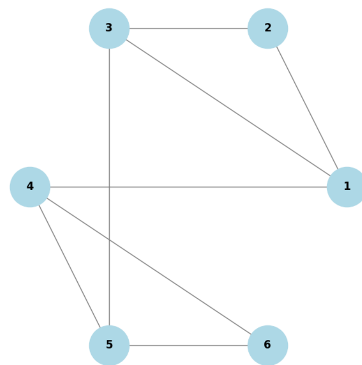

FIGURE 2. A network of six computers.

Consider the graph above, where each node is a computer in the network of 6 computers. If there are n different computers, the maximum number of connections each computer can have is $n-1$. This is the case for any network consisting of n computers (assuming the same conditions as above).

In this scenario, the pigeons are the computers and the pigeonholes are the maximum number of connections. Since the number of computers (pigeons) is greater than the maximum number of connections, there has to be at least two computers in the network with the same number of connections.

Proof. A computer can be connected to at most $5(n-1)$ computers. Each computer is connected to at least one other computer. The number of connections each computer has is denoted by i, with range: $1 \leq i \leq 5$, or a more general version $1 \leq i \leq n-1$. By the Pigeonhole Principle, since there are n computers and $n-1$ connections, $\exists$ at least two computers with the same number of connections. □

Based on this example, one can see how a very simple principle can be used to prove very complex problems in arbitrary sized cases.

3. Why 9 Queens Can't Be Placed on an 8×8 Chess Board

The first step is to identify what the pigeons and pigeonholes are in the context of the problem. In this case, the pigeons are the items we are attempting to place, which are the 9 queens. Since the queens can move along the row, column, or diagonal, we can choose one of these to be the pigeonholes. Using the rows as our pigeonholes, it becomes clear how the principle will be applied.

Proof. A queen can be placed in at most one row. We can divide the board into 8 separate rows. Since there are 9 queens and 8 rows, $\exists$ at least one row with two queens, meaning they attack each other. Therefore, it is not possible to place 9 non-attacking queens on an 8×8 chess board. □

This concept can be generalized to any case with a square chess board $(n \times n)$. Let the number of queens be $n+1$ and the number of rows be n, by the Pigeonhole Principle, there must be at least one row with two queens, meaning they would attack each other. The Pigeonhole Principle allows for a concrete mathematical way to prove that this will be the result in any case, not simply in a certain scenario.

Appendix A. Proof of Pigeonhole Principle

Proof. Assume the Pigeonhole Principle is not true. This would mean there are n pigeons and m pigeonholes with $n > m$, where it is possible that each pigeon would have a unique pigeonhole and no pigeonhole would have more than one pigeon.

Using this assumption, every pigeonhole can have at most one pigeon. Since there are m pigeonholes, the total number of pigeons that can have a unique pigeonhole is at most m. However by the assumption that this principle is false, there are $n > m$ pigeons. This implies that at least one pigeon cannot be placed into a pigeonhole, which contradicts our assumption that every pigeon is assigned to a unique pigeonhole.

Therefore, this concept must be false and the Pigeonhole Principle holds. If $n > m$ then at least one pigeonhole must contain at least two pigeons. □

References

[1] Miklós Bóna. *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory*. 5th. World Scientific, 2024.